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Utility spanning and the Ordered Mean Difference Envelope

by Roger J. Bowden[*]

Abstract

It is shown that the space of optimising portfolios for increasing risk averse utility functions forms a one dimensional manifold, which is the envelope of the ordered mean difference utility generators. The manifold also yields the set of second order stochastic dominant porfolios. The optimising portfolio for any utility function can be obtained by solving the simpler problem for a representative utility generator, which has just two linear segments. This can be done by using linear programming, which in turn can be iterated to trace out the entire efficient set, giving a computationally undemanding way of obtaining the stochastic dominance efficient set. The general efficiency frontier shares the one dimensional property with the mean variance efficient frontier, but unlike the latter, the associated portfolios do not form a convex set, so the two fund theorem of mean variance portfolio analysis does not hold in general.

Keywords

Efficient frontier, equivalent margin, linear programming,ordered mean difference, portfolio analysis,stochastic dominance, utility function, utility generator, utility spanning.

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I Introduction

The limitations of mean variance as a portfolio selection principle are well known, indeed have been the subject of a literature over the past thirty years. It is difficult to render it consistent with Von Neumann Morgenstern choice theory under risk The conventional wisdom is that it will be so if either the utility function is quadratic or the distribution of returns is normal. Objectors have pointed out that quadratic utility functions are necessarily diminishing past a certain point (the limited range problem); that subjective attitudes to low returns should surely be asymmetric with those to high returns; and that quadratic preferences imply that absolute risk aversion is increasing in wealth. Likewise, although the normality requirement was later weakened a little by Chamberlain (1983), the chronic non normality of observed asset returns makes normal returns a shaky foundation. Bigelow (1993) provides a good summary of this literature, as well as an attempt to characterise preferences in a form consistent with the principle.Yet mean variance survives and even flourishes among the practitioners. The problem is that as a practical portfolio solution, there is not much to replace it, even though considerable progress has been made in recent years in developing the stochastic dominance efficient portfolio set as a possible alternative (Bawa et al (1985), Yaari (1987), Levy (1992, 1998), Shalit and Yitzhaki (1994), Post (2001)). Mean variance is readily interpreted, and interpretable to the client. The mean variance efficient set is easily calculated, and indeed in the 'standard' form (short sales allowed), has a convenient one dimensional property, namely that all points along the efficient frontier are one dimensional, in terms of the shadow risk free rate as parameter.

In the present paper it is shown that an almost equally simple expected utility maximising portfolio solution process exists for any risk averse agent, with no limiting assumptions as to return distributions or the nature of preferences. As with mean variance, the solution space is one dimensional, and is easy to calculate - it can be done by standard linear programming with just two linear segments.The developmental tool used to obtain and interpret the solution set is the ordered mean difference (OMD), which can be regarded as casting portfolio theory and empirics in terms of non parametric regression theory, unlike classical stochastic dominance, which considers only the marginal distributions.Originally developed as a fund performance measure (Bowden 2000), OMD theory has been shown to have a number of other potential applications. Post (2001) obtained an OMD schedule as dual to a linear programing approach to stochastic dominance, while Bowden (2002) noted that OMD provides a non parametric diagnostic procedure for the existence of CAPM and the detection of historical pricing inefficiencies in a market equilibrium context. When applied to individual portfolio selection, with an arbitrary utility function, the problem effectively reduces to solving the corresponding portfolio problem for a representative OMD generator, which has a very simple structure.Unlike mean variance, the OMD solution process has a close relationship with second order stochastic dominance. In particular, the OMD generator utility functions can also be regarded as generators for the cumulative areas involved in stochastic dominance, and the efficient sets are identical. In addtion to the specifically portfolio aspects, the paper further develops utility generator and spanning theory.

The underlying programme may thus be said to be one of replacing mean variance analysis by something much more general yet operationally easy to implement. The principal results of the present paper are as follows.

1. The space of utility maximising portolios forms a one dimensional manifold, and takes the form of an OMD efficiency envelope or frontier. A mean variance efficiency frontier is a familiar example of the one dimension aspect. The price of the extra generality to arbitrary utility functions is that the manifold, while remaining one dimensional, is not linear, so that the efficient portolio set is no longer convex and the two fund separation theorem of mean variance no longer holds. Dybvig and Ross (1982) noted that the stochastic dominance portfolio set is not convex and this corresponds

(see point 7 below).

2. The efficient frontier has an envelope relationship with a set of efficient utility generators. A utility generator takes the form of a utility function with just two linear segments like the payoff to the writer of a put option. Russell and Yeo (1988), also Bowden (2000), noted that the expectation of these functions corresponded to the cumulated distribution functions, and the present paper exploits this relationship in a portfolio context. The OMD envelope, or efficiency frontier, is the concave hull of the utility generators.

3. The efficient frontier can easily be located by using linear programming (LP). Post (2002) showed that LP can be used in a stochastic dominance context; see also Young (1998). The OMD efficiency frontier is a simpler LP problem, having just two linear segments, and one does not have to use large scale LP algorithms, a computational convenience to most users.

4. The space of risk averse utility functions is spanned by a set of efficient OMD utility generators. Any risk averse utility function can be written as a weighted sum of the efficient generators, the pattern of weights depending upon the individual's attitudes to risk This also gives us a way of generating utility functions more or less at will, or of smooth approximation in terms of linear or quadratic segments, extending the restricted range of quadratic utility functions, should one desire to use these for any purpose.

5. The spanning property will mean that the individual in his or her portfolio selection will act as though it was done with respect to a single generator, so that for such purposes, any utility function can effectively be replaced by one consisting of just two linear segments as above. In turn, this means that each individual will plot as a particular point along the OMD envelope, and revelation experiments can be designed to find out just where.

6. In the absence of further portfolio constraints such as nonnegativity, the optimal portfolios satisfy the pencil property, which means that when plotted against the optimising portfolio all OMD schedules must cross the horizontal axis together at the same point. This property was noted by Bowden (2002) in a CAPM market equilibrium context. The present paper shows it is true of portfolio selection at an individual level.

7. The OMD envelope is associated with an OMD efficient set, and the latter is identical with the second order stochastic dominance (SSD) efficient set, which is thereby revealed to be one dimensional, with associated features such as the pencil property. As is well known, stochastic dominance involves a correspondence between efficiency according to some utility function (the utility version) and a majorisation relation in terms of the distribution functions (the statistical version). The precise nature of this correspondence depends upon the underlying class of utility functions (e.g. Dybvig (1988 appendix)). We shall take this to be the class of utility functions that are strictly increasing and concave, with marginal utility tending to zero as wealth or returns become infinitely large; augmented by the utility generators where necessary to complete the statistical correspondence. This line of attack reinforces a connection of stochastic dominance with convexity theory noted by Pečarić et al (1992, ch 12). Indeed, many propositions about SSD are provable from convexity theory without much recourse to statistics.

The scheme of the paper is as follows. Section II establishes notation and briefly reviews the generalised functions used in OMD analysis, including the utility generators. It moves on to develop the utility spanning theory. Section III establishes the OMD envelope and its properties. Section IV proves that the optimising solution for a arbitrary risk averse utility function must lie on the resulting frontier, and shows that the OMD efficient set and the SSD efficient set are one and the same. Section IV turns to computation, and shows that LP methods can be used to obtain the efficient frontier. Section V offers some concluding remarks and considers further applications.

II Utility spanning

A Notation

(a) *Returns*. Individual asset returns are written in lower case as r_i for asset $i = 1...n$, or simply r as a representative asset. Often we use the return symbol (r or R) as a convenient shorthand for the relevant asset; thus 'asset r' means the 'asset whose return is r'. No risk free rate is assumed, so this is going to be the standard practitioner portfolio construction. The joint distribution function $\Phi(\mathbf{r})$ of underlying asset returns is taken to be of full rank, and we assume it has a density $\phi(\mathbf{r})$.No assumptions are made about the ambient market equilibrium or disequilibrium, so CAPM or other models of market equilibrium are *not* assumed. Portfolio returns are written in upper case as $R = \sum_i x_i r_i$, where the x_i are collectively a set of portfolio proportions such that $\mathbf{x} \in X = \{\mathbf{x} : \sum x_i = 1\}$, the feasible set of portfolios. The domain D of portfolio returns will usually be taken as R_L<R< ∞, where the lower limit R_L may itself be $-\infty$. For instance, the logarithmic utility function log(1+R) or the square root $\sqrt{1+R}$ will require $R_L > -1$, but others allow an infinite lower return.Often a dummy portfolio return denoted P will be employed. This will have the same domain of definition as R, so that $P_L = R_L$.

(b) *Utility functions*. The investor has a Von Neumann-Morgenstern utility function of the general form U(W,W_0), where W_0 is opening wealth and W is the result from investing that wealth. W and W_0 are related by W = W_0(1+R), where R is the portfolio rate of return. Initial wealth W_0 can enter additionally via wealth effects, likewise past wealth via habit effects, as can other environmental modifiers to the individual's preferences. In what follows, initial wealth W_0 and any other modifiers will be taken as a given datum and notationally suppressed. The resulting utility function is written as U(R). The use of return R to proxy outcome wealth is simply a matter of contextual simplicity and much of the ensuing spanning theory will apply equally to more general contexts.The utility function U will be taken as strictly concave and increasing, so what follows applies to a risk averse investor with increasing utility of money. It is assumed that U(R) is differentiable to order 2 and also that $\lim_{R\to\infty} U'(R) = 0$, i.e. marginal utility tends to zero. Denote by Ψ_u the resulting set of utility functions.

(c) *The ramp function as utility generator.*

For a given return number P, the ramp function $U_P(R)$ is given by

$$U_P(R) = (R-P)SF(P-R),$$

where the step function $SF(P-R)$ can be defined as

$$\begin{aligned} SF(R-P) &= 1 \; ; \; R > P \\ &= \tfrac{1}{2} \; ; \; R = P \\ &= 0 \; ; \; R < P. \end{aligned}$$

Effectively, $U_P(R) = min(R-P, 0)$. However, it is desirable to preserve the step function expression in order to harmonise with operations involving distribution functions. Indeed, all the theory that follows can be expressed in terms of the Temple-Schwartz generalised distribution theory, for which see Lighthill (1959), Antosik et al (1973),or Vilenkin and Klimyk (1995); the same convenience applies to classical stochastic dominance.

The ramp function will play the role of a utility generator. It is sketched in figure 1 for different values of the parameter P; the latter may be called the focal point or node of the generator.Viewed as a utility function, the risk aversion of the generator $U_P(R)$ relative to a fixed return distribution diminishes as P increases from P to P$'$. It will be observed that for fixed P, $U_P(R)$ is concave in R, though not strictly so. It is not technically differentiable at $R = P$. However, we can define the derivative in the sense of Temple Schwartz theory as $U'_P(R) = 1 - SF(R-P) = SF(P-R)$. The collection of generators $\{U_P(R); P_L < P < \infty\}$ will constitute a useful completion of the chosen set of utility functions as specified under (b) above.

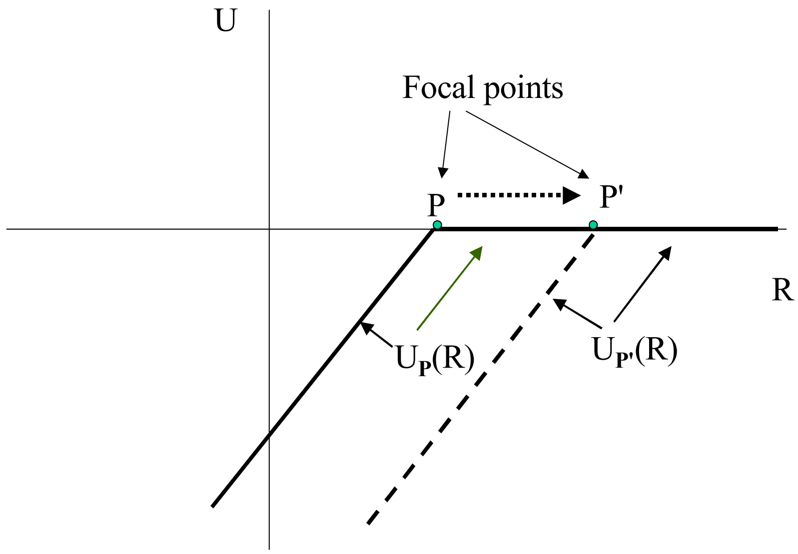

Figure 1: The OMD utility generator

Denote by $F(R;\mathbf{x})$or just $F(R)$ the distribution fucntion of portfolio returns for a given portfolio $\mathbf{x}$, and let $f(R)$ be the corresponding density. The utility generators are connected with stochastic dominance theory via the relationship

$$E[U_P(R)] = -\Im(P) = -\int_{P_L}^{P} F(R)dR, \qquad (1)$$

#

where the function $\Im(P)$ will be recognised as the stochastic dominance cumulator, namely the area underneath the distribution function up to the chosen point P. Also useful is the running or truncated mean function defined by

$$\mu(P) = \frac{1}{F(P)} \int_{P_L}^{P} f(R)dR,$$

which is the average over values of R less than or equal to the given value P.

B Utility spanning

Although the utility generators U_P(R) are themselves non differentiable and of very simple form, they have some useful spanning properties. Under mild regularity conditions, we can show that the expected utility of any otherwise arbitrary utility functions U(R)∈ Ψ_u can be written as a weighted average of the utility generators:

$$E[U(R)] = \alpha_0 + \alpha_1 \int_{P_L}^{\infty} w(P)E[U_P(R)]dP \; ; w(P) \geqslant 0, \int_{P_L}^{\infty} w(P)dP = 1, \qquad (2)$$

#

where α_0 and α_1 >0 are some constants, allowable under Von Neumann Morgenstern choice theory.This turns out to be true, with the choice $w(P) \propto -U''(P)$.In turn, this implies (in section III) that the portfolio theory of Ψ_u reduces to that for the representative generator. However, we shall first demonstrate the spanning property.

Lemma 1

Suppose $P_L < R \leqslant P_a$, some finite upper limit. Then relative to some fixed point R_0 such that $P_L < R_0 \leqslant P_a$,

$$U(R) - U(R_0) = U'(P_a)(R - R_0) + \int_{P_L}^{Pa} (-U''(P))[U_P(R) - U_P(R_0)]dP \ . \qquad (3)$$

#

Proof

Simple integration by parts shows that

$$U(R) = U(P_a) + U'(P_a)(R - P_u) + \int_{P_L}^{Pa} (-U''(P)U_P(R)dP \ .$$

Setting $R = R_0$ and subtracting yields the desired result (3).

□

In a portfolio context, the return $R = R(\mathbf{x})$ depends upon the chosen portfolio $\mathbf{x}$. Sometimes we shall emphasise this by writing the expected utility as $E[U(R;\mathbf{x})]$, but it will be taken in what follows that this dependence holds even where not explicitly annotated as such

Proposition 1

Suppose the utility function $U \in \Psi_u$ and distribution of returns are such that both $E[R]$ and$E[U(R)]$ exist, uniformly in $\mathbf{x} \in X$. Then

$$E[U(R)] \ = \alpha_0 + \alpha_1 \int_{P_L}^{\infty} w(P)E[U_P(R)]dP \ ; w(P) \geqslant 0, \int_{P_L}^{\infty} w(P)dP = 1,$$

where α_0 and $\alpha_1 > 0$ are constants and $w(P) \propto -U''(P)$.

Proof

$$E[U(R) - U(R_0)] \ = \int_{P_L}^{P_a} (U(R) - U(R_0))f(R)dR + \int_{P_a}^{\infty} (U(R) - U(R_0))f(R)dR.$$

From the given hypotheses, the second term on the RHS must tend to zero uniformly in $\mathbf{x}$ as $P_a \to \infty$. Substituting expression (3) into the first RHS term and letting $P_a \to \infty$ gives the required result, noting that $U(R_0)$ and $U'(R_L)$ will just amount to constants.

□

Remarks:

(a) The condition that for any fixed portfolio $\mathbf{x}, E[R;\mathbf{x}]$ or $E[U(R;\mathbf{x})]$ converge uniformly in $\mathbf{x}$ should not be taken as a necessary limitation on the portolio mean, which can indeed be unbounded over different portfolio choices $\mathbf{x}$, just as for mean variance analysis.It refers instead to the convergence of the improper intergrals used to obtain the expected values, and is no more than what is required by standard portfolio theory.

(b) The spanning condition as given relates to expected utility. However by differentiating expression (3) we see that pointwise in R,

$$U'(R) = \int_{P_L}^{\infty} (-U''(P))U'_P(R)dP \qquad (4),$$

#

which means that the spanning property also applies to marginal as well as expected utility.Property (4) can be used to simplify a number of the proofs of OMD properties in Bowden (2000).

III Generator solution properties

A Properties of expected generator utility

Expected generator utility is defined for portfolio **x** as

$$V(P;\mathbf{x}) = E[U_P(R;\mathbf{x})] = -\Im(P;\mathbf{x}) \; , \qquad (5) \qquad \#$$

In what follows, we shall often refer to the function $V(P;\mathbf{x})$ as the 'SSD cumulator', or just 'cumulator', for the given portfolio, taking the negative sign as understood. For fixed **x**, the stochastic dominance cumulator equivalence (5) shows us that $V(P;\mathbf{x})$ will be decreasing, such that $0 > V(P,\mathbf{x}) > -\infty$, and C^2 in P. For fixed P, $U_P(R)$ is a concave function of $R = \mathbf{x}'\mathbf{r}$, which is linear in elementary returns **r**. Hence it will be true that $V(P;\mathbf{x})$ will be concave in **x** .The following result shows that at least for $P \neq 0$, it is strictly concave.

Proposition 2

*Suppose the distribution $\Phi(\mathbf{r})$ of returns is of full rank. If $P \neq 0$, the Hessian H of $V(P;\mathbf{x})$ with respect to **x** is negative definite, everywhere in X.The Hessian is negative semidefinite for $P = 0$.*

Proof: See Appendix A.

□

Corollary

If $P \neq 0$, then the portfolio optimising solution is unique and a parametric function $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(P)$ of P. If in addition $V(P;\mathbf{x})$ is C^2 in P, then $\mathbf{x}(P)$ is C^1.

Proof

The uniqueness follows from the strict concavity of $V(P;\mathbf{x})$ with respect to **x**, and the convexity of X. The existence and C^1 property of $\mathbf{x}(P)$ follows from the implicit function theorem and the first order optimising conditions.

□

Remark: The behaviour at P=0 is not troublesome; at worst, it is associated with the possibility of a point of inflexion; see Bowden (2003,figure 2) for an example in the context of hedging theory.

B The OMD envelope

We can now consider what happens to the optimising generator solutions as the focus P varies.

Definition:

The OMD envelope, or the OMD efficient frontier, is the function $v(P)$ defined by

$$v(P) = V(P;\tilde{\mathbf{x}}(P) = E[U_P(R); R = \tilde{\mathbf{x}}(P)'\mathbf{r}] \; . \qquad (6) \qquad \#$$

It follows from the envelope theorem of concave programming[1] footnote that $v(P)$ is the mathematical envelope of the functions $V(P,\mathbf{x})$. In other words,

$$v(P) \geqslant V(P;\mathbf{x}),\, all\, \mathbf{x}; \qquad (7a) \qquad \#$$

$$v'(P) = \frac{\partial V(P;\tilde{\mathbf{x}})}{\partial P} \; . \qquad (7b) \qquad \#$$

Figure 2 shows how things must look. Let $\mathbf{x}_0 = \tilde{\mathbf{x}}(P_0)$ be the optimum portfolio at $P = P_0$, so that $v_0 = V(P_0,\tilde{\mathbf{x}}(P_0))$. For any other portfolio proportions $\mathbf{x}_1$ say, we must have $V(P_0,\mathbf{x}_1) < v_0$. The same property will be true of portfolio $\mathbf{x}_0$ at $P = P_1$ in relation to $v_1 = V(P_1,\tilde{\mathbf{x}}(P_1))$ .The individual schedules $V(P;\mathbf{x})$ have slope asymptotically -1, as does the envelope itself In mathematical terms (Rockafeller (1970 ch1 §5)), the OMD envelope is the concave hull of the utility generators, and its epigraph is the union of those for the individual generators.

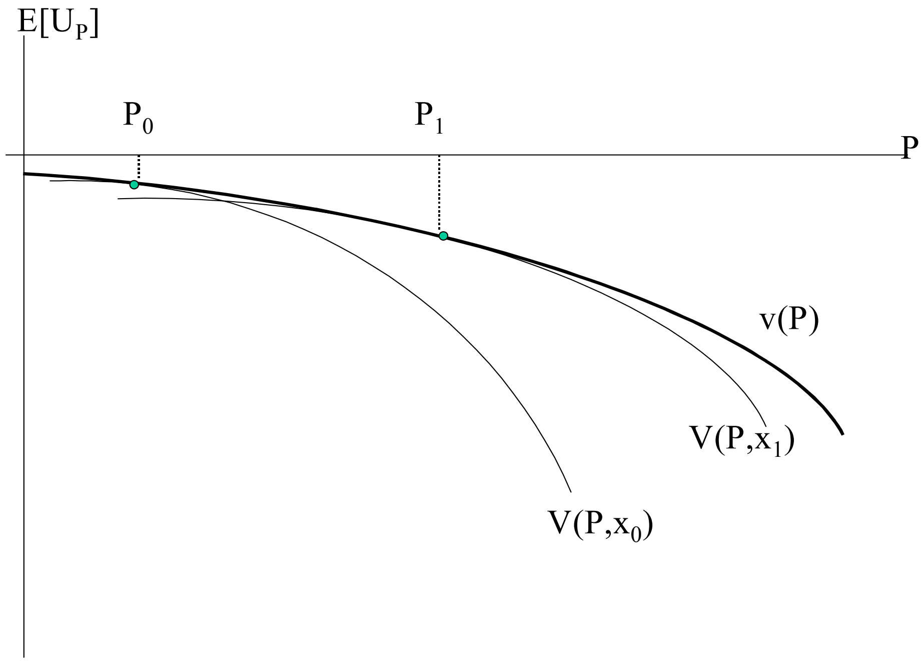

Figure 2: The OMD envelope $v(P)$

Save in one special case, the OMD envelope does not in itself represent the expected utility from any one portfolio That is, there does not exist a portfolio $\mathbf{x}_d$ such that $v(P) = E[U_P(R; \mathbf{x}_d)]$. One could ask what would happen if the portfolios over differing P (e.g. $\mathbf{x}_0$ and $\mathbf{x}_1$) were insensitive to P. In this case the local portfolio plots of figure 2 would run together and merge with $v(P)$. The interpretation is that the common portfolio $\mathbf{x}_d$ would optimise $E[U_P(R)]$ for every value of P. This is the absolute stochastic dominance criterion.

Figure 3 illustrates the OMD envelope in terms of second order stochastic dominance, making use of expression (5) above. Figure 3 is effectively figure 2 rotated 180^0 about the horizontal axis. In this form it will be familiar as the standard SSD construction involving the cumulated distribution functions, in this case of the portfolio returns.The OMD envelope is that of the SSD cumulators applied to the respective optimal $\tilde{\mathbf{x}}(P)$ portfolios. The nature of the construction suggests that the OMD envelope will generate the SSD efficient set, and this turns out to be true.

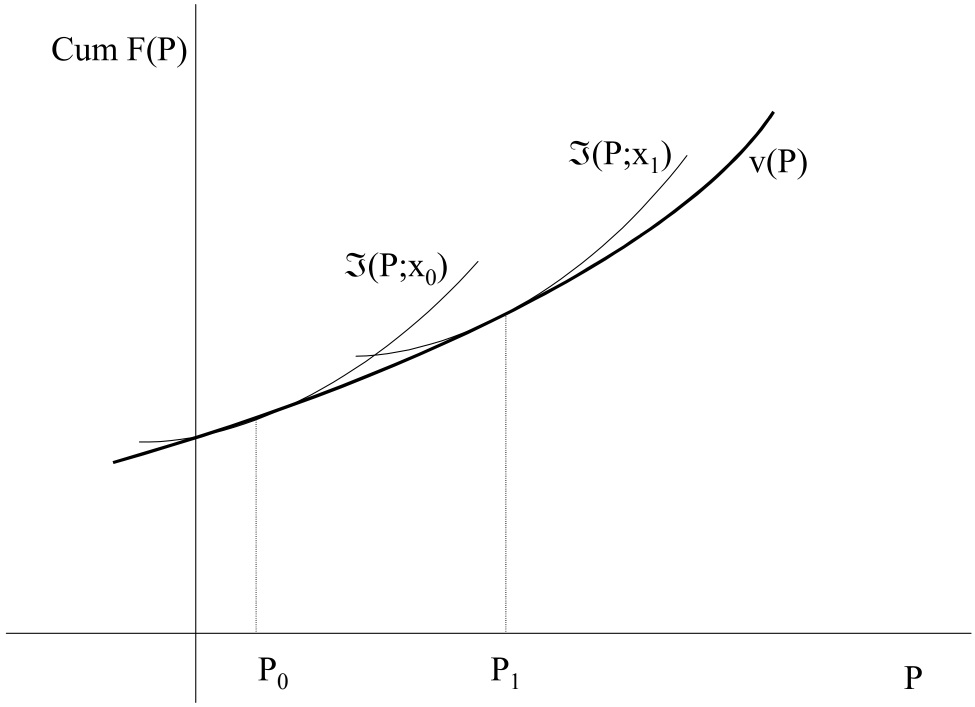

Figure 3: The OMD envelope in stochastic dominance terms

C The pencil property

The ordered mean difference schedule $\tau_i(P)$ (Bowden 2000) for return r_i against a benchmark R is defined by

$$\tau_i(P) = \frac{E[(r_i - R)U'_P(R)]}{E[U'_P(R)]} = E_P(r_i - R) \,, \qquad (8) \qquad \#$$

where the operator $E_P(.)$ denotes the running mean difference obtained by first ordering the paired observations by increasing R values, then for any P, taking the average difference corresponding to values of R less than or equal to P.

For a given P_0, the first order conditions for the generator problem reduce to $E[(r_i - R)U'_{P_0}(R)] = 0,$ for all i.Measured against the optimum portfolio $\tilde{R}_0 = \tilde{\mathbf{x}}(P_0)'\mathbf{r}$ as benchmark,this means that all OMD schedules $\tau_i(P)$cross the horizontal axis at a common point, namely P_0. We call this the pencil property. Figure 4 illustrates. The OMD schedules may not necessarily be monotonic as illustrated and could have individual multiple crossing points[2].By a process of extension, all points along the OMD envelope must share the pencil property. The OMD efficiency frontier is generated by portfolios $\tilde{R}$ such that when measured against $\tilde{R}(P)$, all OMD schedules cross at the common point P.

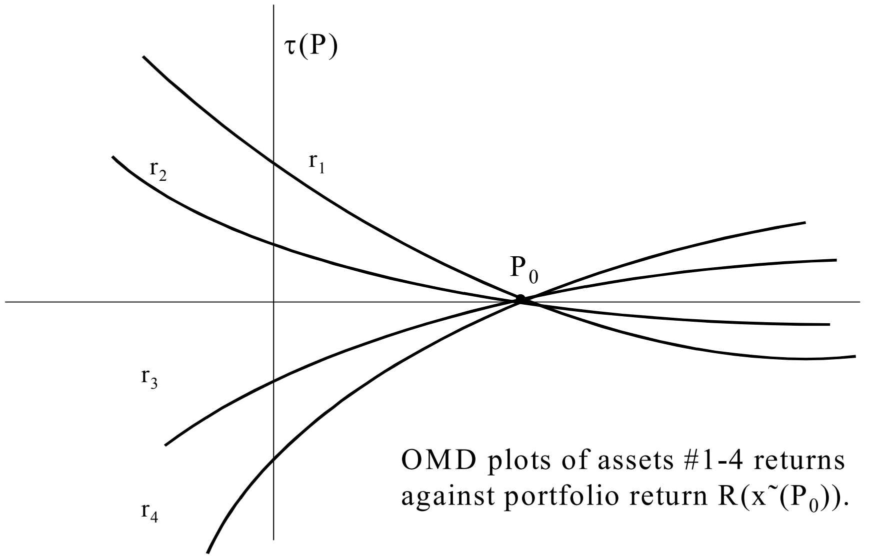

Figure 4: Pencil property of optimal generator solutions

D Inefficient portfolios

Inefficient portfolios will be those where the cumulators V(P,**x**) plot entirely inside the OMD frontier, illustrated in figure 5 below. There exists no point of tangency, so that it is not true that **x** = $\tilde{\mathbf{x}}(P_0)$, for some P_0 . However, given an arbitrary portfolio **x** it is always possible to bound the corresponding cumulator $V(P,\mathbf{x})$ by means of of a weighted average of the efficient cumulants $V(P,\tilde{\mathbf{x}}(P_0))$.

To do this, we introduce the idea of a weighting function $w_m(P)$ analogous to a probability density with $w_m(P) \geq 0$ and $\int_D w_m(P)dP = 1$.Consider a weighted sum of the form

$$F_m(R) = \int_{P_0} w_m(P_0)F(R;\tilde{\mathbf{x}}(P_0))dP_0 \ ,$$

in which $F(R;\tilde{\mathbf{x}}(P_0))$ is the distribution generated by the efficient portfolio at P_0 .Evidently $F_m(R)$ is a mixed distribution, though not necessarily one corresponding to any single portfolio It can instead be regarded as generated by a variable porfolio, in the following sense. The cumulator for the mixture may be written as

$$V_m(P) = \int_{P_0} w_m(P_0)V(P,\tilde{\mathbf{x}}(P_0))dP_0 \,.$$

As a function of P_0, for fixed P, the element $V(P,\tilde{\mathbf{x}}(P_0))$ is concave in P_0 with a unique maximum at $P_0 = P$. Using the second mean value theorem of integral calculus applied to the right hand integral, we can therefore write

$$V(P) = V(P,\tilde{\mathbf{x}}(\pi(P)) \quad ;\pi(P) \leq P. \qquad (9) \qquad \#$$

We can interpret this as saying that the cumulator constructed from any mixture can always be regarded as a cumulator generated by a variable portfolio, if not a fixed one. By altering the weight function we can alter the mapping defined in (9) at will.

Now let the cumulator $V(P,\mathbf{x})$ be generated by some arbitrary portfolio $\mathbf{x}$. Given any return number P, we have $V(P,\mathbf{x}) < V(P,\tilde{\mathbf{x}}(P))$, and from the continuity of $\mathbf{x}(P)$ there must exist $P_0 < P$ such that $V(P,\mathbf{x}) = V(P,\tilde{\mathbf{x}}(P_0))$. Figure 5 illustrates,with the tangency point at A. Writing $P_0 = \pi(P)$, we must have $\pi(P) \leq P$, as above. Thus we can write

$$V(P,\mathbf{x}) = V(P,\tilde{\mathbf{x}}(\pi(P))) \ ;\pi(P) \leq P.$$

This is again of variable portfolio form, with the portfolios drawn from the efficient frontier.We can always find some weighting function $w_x(P)$ with associated mapping $\pi_x(P) \geqslant \pi(P)$, and by means of this bound the given generator cumulator by a weighted sum of the efficient portfolio cumulators. This may be summarised as follows.

***Proposition* 3**

Let portfolio $\mathbf{x}$ *be arbitrary. Then we can always find a weighting function* $w_x(P_0)$, *in general depending upon* $\mathbf{x}$, *such that for any return number* P,

$$V(P,\mathbf{x}) \leq \int_{P_0} w_x(P_0)V(P,\tilde{\mathbf{x}}(P_0))dP_0. \qquad (10)$$ #

Remark: If portfolio $\mathbf{x}$ already belongs to the OMD efficient frontier, then $\mathbf{x} = \tilde{\mathbf{x}}(P_{0x})$ for some P_{0x}. In this case, $w_x(P_0) = \delta(P_0 - P_{0x})$, the Dirac delta function[3] centred at P_{0x} , and equality holds in (10).

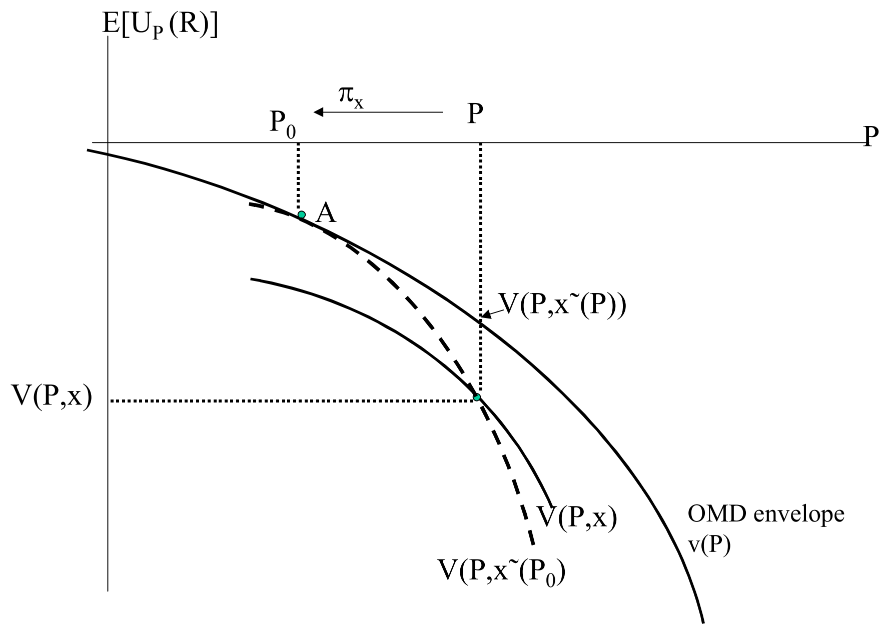

Figure 5 Inefficient portfolio relationship to frontier

IV General utility functions

A The envelope as optimal

We can now exploit the above generator/cumulant properties to derive the principal theoretical result of the paper, namely that the OMD envelope must contain the solution for any $U \in \Psi_u$. Suppose that $U(R)$ is such a utility function , and let $\mathbf{x}_*$ be its optimising portfolio solution. The theorem that follows shows that this portfolio must plot somewhere along the OMD envelope. In other words, there must always exist a return number P_* such that $\mathbf{x}_* = \tilde{\mathbf{x}}(P_*)$.

***Theorem* 1**

For any utility function $U \in \Psi_u$ *, the optimum portfolio* $\mathbf{x}_* = \tilde{\mathbf{x}}(P_*)$ *for some return number* P_* *.The latter is explicitly given by*

$$P_* = \arg\max_{P_0} \int_P w_u(P) V(P, \tilde{\mathbf{x}}(P_0)) dP,$$

where the weighting function $w_u(P)$ is proportional to $U''(P)$, the second derivative of the given utility function at $R = P$.

Proof

The objective is to maximise $E[U(R(\mathbf{x}))]$, subject to the adding up constraint $1'\mathbf{x} = 1$. Using expression (2) of section II we can replace this with the equivalent problem

$$\max_{\mathbf{x}} \{V(\mathbf{x}) = \int_P w_u(P) V(P, x) dP\} .$$

Suppose P_*=arg $\max_{P_0} \int_P w_u(P) V(P, \tilde{\mathbf{x}}(P_0)) dP$; that is, $\tilde{\mathbf{x}}(P_*)$maximises $V(\mathbf{x})$ over $\mathbf{x} \in X_{OMD}$.For an arbitrary portfolio $\mathbf{x}$, we have

$$\begin{aligned} V(\mathbf{x}) &= \int_P w_u(P) V(P, \mathbf{x}) dP \\ &\leqslant \int_P \int_{P_0} w_u(P) w_x(P_0) V(P, \tilde{\mathbf{x}}(P_0)) dP_0 dP \qquad \textit{from proposition } 3 \\ &= \int_{P_0} w_x(P_0) \{\int_P w_u(P) V(P, \tilde{\mathbf{x}}(P_0)) dP\} dP_0 \\ &\leq \int w_u(P) V(P, \tilde{\mathbf{x}}(P_*)) dP \\ &= V(\tilde{\mathbf{x}}(P_*)) . \end{aligned}$$

Thus given the optimising portfolio $\mathbf{x}_*$ is unique, it must correspond with $\tilde{\mathbf{x}}(P_*)$.
□

Corollary

The OMD schedules of the elementary assets have the pencil property with respect to the optimum portfolio R_*.

B The OMD and SSD efficient sets

As mentioned in the introduction, stochastic dominance comes in different specifications, and the stochastic dominance portfolio set will reflect the distinctions. However, we shall take the second order stochastic dominance efficient set of portfolios X_{SSD} as being the set of portfolios undominated in Ψ_u. Thus $\mathbf{x} \in X_{SSD}$ if and only if there does not exist another portfolio $\mathbf{x}_u$ such that

$V(\mathbf{x}_u) > V(\mathbf{x})$ for any utility function u $\in \Psi_u$. It follows from the preceding development that $X_{SSD} = X_{OMD}$; the OMD efficient set and the SSD efficient set are one and the same.

This means that we can locate the SSD set with methodology that locates the OMD envelope. Moreover, the SSD set is recognised as being one dimensional: the solution to every portfolio maximisation problem must lie in this set and likewise be one dimensional. In addition, the elements of the SSD set all exhibit the pencil property and other properties associated with the OMD envelope. For this reason we can refer to the OMD envelope simply as the efficient set, with no further qualification.

C Tests for efficiency

Given a specific portfolio one might wish to test whether it belongs to the efficient set. The foregoing development suggests two criteria for a given portfolio $\mathbf{x}_g$ to be efficient:

(a) Compute the OMD schedules for the invidual assets and check whether these cross at any common point, or equivalently whether $\|\boldsymbol{\tau}(P;R_g)\| = 0$, for any value of P.

(b) Plot the function $V(P;\mathbf{x}_g) = E[U_P(R_g)]$. If this lies wholly inside the OMD envelope, then the given portfolio cannot be efficient.

IV Computation of the efficient set

A Sample aspects

In practice one will have available a time series of sample observations for $t = 1,2,...T$ on the elementary returns, and the efficient set is found by maximising the sample expected generator utilities for sucessive values of P. The relevant sample statistics are:

1. Sample expected utility for portfolio R:

$$\hat{E}[U_P] = \frac{1}{T}\sum_{t=1}^{T}(R_t - P)SF(P - R_t) \ . \qquad (11a) \qquad \#$$

2. Sample OMD ordinate at P for asset i :

$$\hat{\tau}_i = \hat{\tau}_i(P) = \frac{1}{N(P)}\sum_{t=1}^{N(P)}(r_{it} - R_t) \ ; N(P) = \{\#t : R_t \leqslant P\} \ . \qquad (11b) \qquad \#$$

The immediate empirical tasks are to find the portfolio $\tilde{\mathbf{x}}(P)$ associated with the OMD generator $U_P(R)$ and to graph the associated OMD envelope or efficiency frontier $v(P)$ as P varies. In principle, a variety of techniques is available to solve the portfolio problem. Techniques of nonlinear programming do not appear to work well, especially those based on the Hessian, though convergence outcomes have been obtained with conjugate gradient methods, provided the starting point is not too far from the optimum. Hower, it turns out that linear programming methods are applicable, as the ensuing development shows, and experience has shown that this works well, even for quite large data sets

B The Linear programming solution

Figure 6 shows the linear programming nature of the problem. For a representative observation, the portfolio mean is $\mathbf{x}'\mathbf{r}_t$ and the utility element is $\theta_t = (\mathbf{x}'\mathbf{r}_t - P)SF(\mathbf{x}'\mathbf{r}_t - P)$. We must have $\theta_t \leqslant 0$ and $\theta \leqslant \mathbf{x}_t'\mathbf{r}_t - P$. As such θ_t could in principle lie anywhere within the feasible set (shaded boundary), but we shall see that at the LP optimum, it must lie on one or other of the two boundaries, as $U_P(R_t)$ requires.

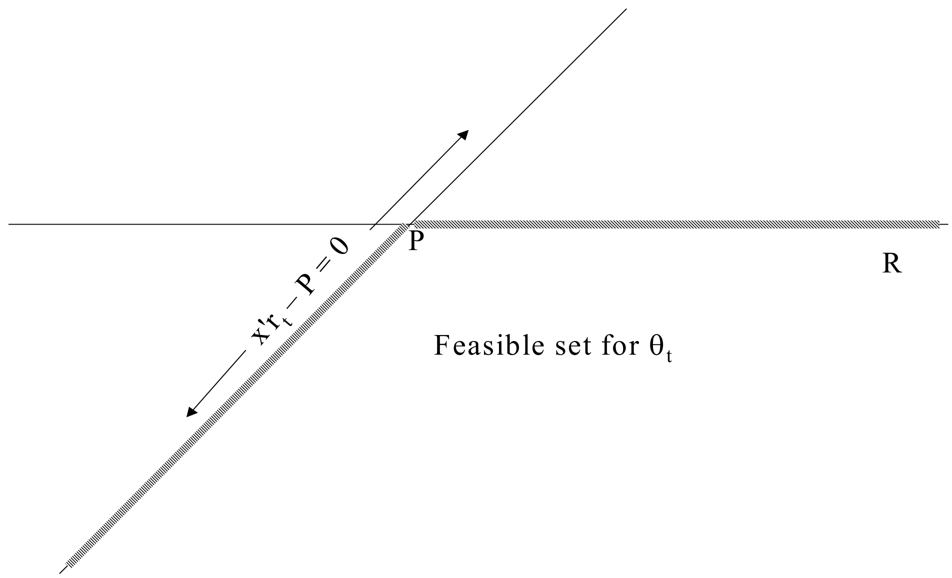

Figure 6: OMD linear programming feasible set

The full LP primal can be written as follows. Let $A = ((r_{t,i}))$ denote the Txn data matrix of observations on the asset returns.

$$\max_{\mathbf{x},\boldsymbol{\theta}} \ eu = \frac{1}{T}\mathbf{1}'\boldsymbol{\theta} \ \textit{subject to}: \qquad (12) \qquad \#$$

$$\boldsymbol{\theta} \leqslant \mathbf{0} \qquad (\boldsymbol{\lambda}_1)$$

$$\boldsymbol{\theta} - A\mathbf{x} \leqslant -P\mathbf{1} \qquad (\boldsymbol{\lambda}_2)$$

$$\mathbf{1}'\mathbf{x} = 1 \qquad (\lambda_3)$$

$$\mathbf{x} \ \textit{free}.$$

The dual to problem involves a dual vector partitioned conformably with (12) as $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}_1', \boldsymbol{\lambda}_2', \lambda_3)$, and reads:

$$\min_{\boldsymbol{\lambda}} \ eu = -P\boldsymbol{\lambda}_2'\mathbf{1} + \lambda_3 \ \textit{subject to}: \qquad (13) \qquad \#$$

$$\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 \geqslant \frac{1}{T}$$

$$-A'\boldsymbol{\lambda}_2 + \lambda_3 1 \geqslant \mathbf{0}$$

$$\boldsymbol{\lambda} \geqslant \mathbf{0} \ .$$

It is clear from the first constraint of (13) that at least one of every pair $\lambda_{1t}, \lambda_{2t}$ must be strictly positive at the optimum, and hence that θ_t lies on one or other of the branch boundaries of the feasible set as illustrated in figure 6. Hence the utility element $U_P(R_t)$is correctly captured.

Figure 7 illustrates an empirical OMD efficiency frontier, plotted at intervals of 0.025% on an annual return basis The data is a teaching set from Lorimer and Rayhorn (2002) consisting of 59 monthly returns from 1995-1999 on 14 stocks chosen as representative of the industrial coverage in the S&P 500. The LP routine used is the revised simplex algorithm DLPRS from the Fortran IMSL(math) library. The empirical OMD schedule generally conforms to the sketch of figure 2 or 5, except that it has a lower zero deriving from a finite data set.Note that its slope is asymptotically -1, as the theory requires.

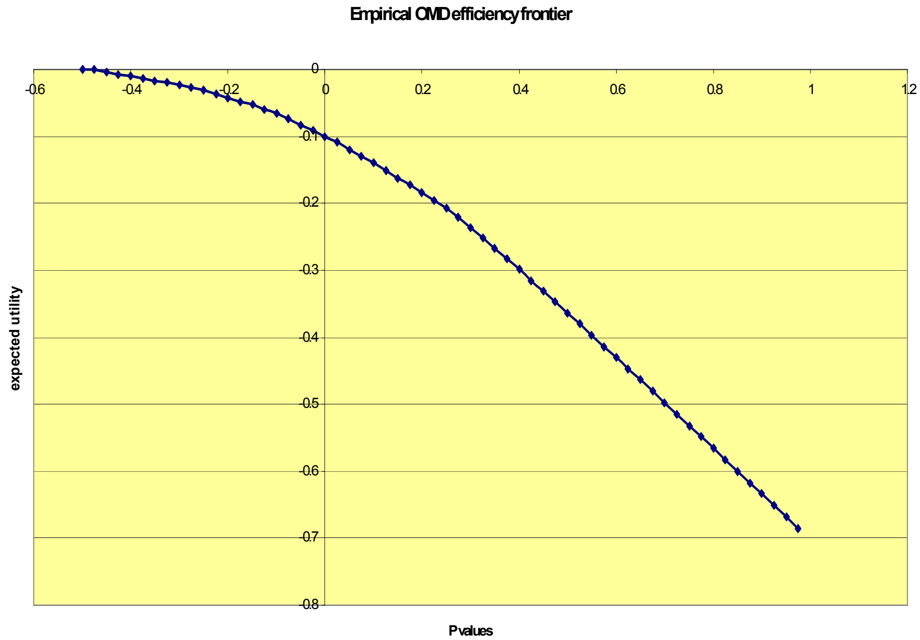

Figure 7: Empirical OMD efficiency frontier

The implication is that faced with this data set of returns, the efficient portolio $\mathbf{x}_*$ for every risk averse investor would lie somewhere along this one dimensional manifold, in the sense that there would be some point P_* such that the associated efficient portfolio $\tilde{\mathbf{x}}(P_*) = \mathbf{x}_*$.

The OMD envelope can be used as a quick internally generated check for the existence of universal stochastic dominance. Such a single portfolio, if it exists, will have the property that it is optimal for all P. On the other hand if the optimal generator portolios vary with P, this will indicate that an absolute SSD relationship does not exist among the asset returns of the chosen data set. The computations associated with Figure 7 indicated that the optimal portfolios did differ systematically as P varied, so that there is no stochastic dominance in the Lorimer-Rayhorn data.

V Concluding remarks

A Applications

The one dimensional nature of the OMD efficient frontier can be exploited in a number of ways.

(a) It is identical with the stochastic dominance frontier associated with Ψ_u. Solving the associated LP problem as in the preceding section is an easy way of obtaining the SSD efficient set, undemanding in its computational aspects.

(b) Given a specific utility function, one can proceed to replace the full n dimensional problem $max_{\mathbf{x}}E[U(R)]$with search based on just the one dimension. Solve the LP problem to get the portfolio associated with each P. Insert $\tilde{\mathbf{x}}$(P) into the given expected utility function and repeat along the P dimension.

(c) A further potential application is where one does not know the agent's utility function in advance. In investment advisory work, the mean variance frontier is commonly used as a portfolio selection revelation device. The client is presented with a range of alternative portfolios along the frontier, together with the corresponding means and standard deviations, as representing the trade off between apparent expected reward and risk He or she selects the particular point on the basis of a preferred combination of expected reward with risk. Given the well known limitations of mean variance analysis, it might be possible to devise a revelation experiment to locate the preferred OMD efficient portfolio, which is not subject to these limitations. Appendix B sketches out a possible revelation experiment for this purpose.

(d) In some circumstances, one is interested to know whether a preferred portfolio strategy is sensitive to different choices of the underlying utility fucntion. An example arises in hedging. If returns of the portfolio to be hedged and the proposed hedge instrument are jointly normal, then a linear hedge exists that remains invariant to the manager's utility function. In general, however, return distributions are not normal, so the issue arises as to whether a given hedge ratio (e.g. that derived from OLS) remains optimal under different utility functions. In this case one can exploit the semi parametric nature of the utility generators, and evaluate the optimal hedge ratio as a function of P, thereby deriving conditions under which invariance will hold (Bowden 2003).

B Comparative properties

Empirically, it is possible to test whether any given point along the mean variance efficient frontier is also OMD or SSD efficient (just evaluate the equivalent margins and check the pencil property). For the the Lorimer Rayhorn data used in section IV the answer is negative.

However, mean variance analysis does have one useful property, namely that the efficient set is convex. Thus if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two MV efficient portfolios, so is $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$. The convexity arises because the MV efficient set is linear. Indeed, if $\boldsymbol{\mu}$ and Σ are respectively the mean and covariance matrix of security returns, then $\mathbf{x} \propto \Sigma^{-1}(\boldsymbol{\mu} - \rho\mathbf{1})$, where ρ is an actual or notional risk free rate that serves parametrically to trace out the MV efficient frontier. In other words, we can write $\mathbf{x} = \mathbf{x}(\rho)$, tracing out a manifold which is not only one dimensional but also linear affine. It is the convexity property that leads to the two fund separation theorem, which says that any efficient portfolio can be generated from just two ρ such as the risk free asset and the market portfolio in a capital market equilibrium context.

Like the MV frontier, the OMD efficient frontier forms a one dimensional manifold, but it is no longer necessarily linear; nonlinearity is the price paid for the extra generality. Correspondingly, convexity is no longer a general property, and the two fund theorem will not hold, in this respect reflecting the findings of Dyvbig and Ross (1982).

C Incidental applications

The utility spanning theory of the present development can itself be applied to derive flexible representations of utility preferences. The expected utility spanning properties of section II indicate that for all practical purposes we can consider utility functions made up of weighted combinations of the OMD generators, i.e. of the form

$$U(R) = \int_{P_L}^{\infty} w(P)U_P(R)dP \ ; w(P) \geqslant 0, \int_{P_L}^{\infty} w(P)dP = 1,$$

for some suitable choice of the weight function $w(P)$. For example,we could let $w(P)$ be the ordinates of some probability density function. This would result in bounded marginal utility, which may or may not be viewed as a limitation. However, it does enable one to draw on a range of flexible representations associated with distributions such as the Weibull, or extreme value families. It also ties in quite nicely with the interpretation of the risk averse individual as made up of an assemblage of 'gnomes', one for each value of P. A more risk averse investor would have more gnomes in low values of P. Another possibility is that one can build utility functions up from elementary components such as linear or quadratic segments, which may be useful in reducing the expected utility maximisation to a linear or quadratic programming problem. This may be useful in correcting an inherent defect of a simple quadratic utility function, namely that it diminishes in wealth after a certain point. Grafting can be done with the aid of improper probability distributions of the kind used in Bayesian analysis.

The device of a discrete probability density $\{w_k; k = 1,2,..K\}$ can be used to approximate a desired utility profile, perhaps initially drawn by freehand, by a series of linear segments. Thus for a series U(R_i), 1=1,2,...,N, of given ordinates and any given number K of segments, one can set up a regression model:

$$U_i = U(R_i) = a + \sum_k w_k U_{P_k}(R_i)$$

and find the best fitting weights w_k by OLS or some form of weighted least squares.Trial plots show that the approximation becomes quite smooth after a surprisingly small number of segments.

Appendices

A Proposition 2

Suppose the distribution $\Phi(\mathbf{r})$ *of returns is of full rank. If* $P \neq 0$, *the Hessian* H *of* $V(P;\mathbf{x})$ *is negative definite*, *everywhere in X.The Hessian is negative semidefinite for* $P = 0$.

Exposition of the proof is made easier by calling on some generalised function calculus. The step function was defined in section IIAc of the text. In addition we make use of the Dirac delta function, defined as

$$\delta(R-P) = 0 \;; R \neq P$$
$$\int_{P_L}^{\infty} \delta(R-P)dR = 1.$$

The unit step function may be viewed as a limiting case of a continuous distribution function such as the normal distribution with mean P and a variance that tends to zero, and the Dirac delta as the corresponding limiting density. The Dirac delta has the ‘filtering property’ that for any suitably smooth function $f(R)$, $\int_G f(R)\delta(R-P)dR = f(P)$ and $\int_G f(R)\delta^{(n)}(R-P)dR = (-1)^n f^{(n)}(P)$ for the derivatives. Similarly, if $f(R)$ is a probability density, $\int_G SF(P-R))f(R)dR = F(P)$, the corresponding distribution function. Lighthill (1959) has a convenient summary of such results.Notice also that $U'_P(R) = SF(P-R) - (R-P)\delta(P-R)$, and if $f(R)$ is a probability density $E[U'_P(R)] \; = F(P)$.

Proof of Proposition 2

Expected generator utility is given by

$$\begin{aligned} V(P;\mathbf{x}) &= E_{\mathbf{r}}[(R-P)SF(P-R)] \\ &= \int \dots \int_{G(r)} (\mathbf{x}'\mathbf{r} - P)SF(P-\mathbf{x}'\mathbf{r})\phi(\mathbf{r})d\mathbf{r} \;, \end{aligned}$$

where ϕ is the density of returns $\mathbf{r}$, which have domain G($\mathbf{r}$). Using the rules for differentiating generalised distributions,

$$\begin{aligned} \frac{\partial V}{\partial \mathbf{x}} &= \int \dots \int_{G(r)} \mathbf{r} SF(P-\mathbf{x}'\mathbf{r})\phi(\mathbf{r})d\mathbf{r} \\ H = \frac{\partial^2 V}{\partial \mathbf{x}\partial \mathbf{x}'} &= -\int \dots \int_{G(r)} \mathbf{r}\mathbf{r}'\delta(P-\mathbf{x}'\mathbf{r})\phi(\mathbf{r})d\mathbf{r} \;. \end{aligned}$$

Let $\mathbf{z}$ be any non zero vector. Then

$$\mathbf{z}'H\mathbf{z} = -\int \ldots \int_{G(r)} (\mathbf{z}'\mathbf{r})^2 \delta(P - \mathbf{x}'\mathbf{r})\phi(\mathbf{r})d\mathbf{r} \ ,$$

confirming that H is at least negative semidefinite. For $\mathbf{z}'H\mathbf{z}$ to be zero would require the following to be true:

$$x_1 r_1 + \ldots + x_n r_n = P \qquad (A1a)$$
$$z_1 r_1 + \ldots + z_n r_n = 0 \ , \qquad (A1b)$$

with (A1b) holding almost everywhere in $\mathbf{r}$.

Suppose P ≠ 0.Choose a non zero element of **z**; suppose this is z_n .Multiply (A1a) by z_n, (A1b) by x_n and subtract. We end up with

$$\tilde{z}_1 r_1 + \ldots + \tilde{z}_{n-1} r_{n-1} = z_n P \ , \qquad (A2)$$

where $\tilde{z}_i = x_i z_n - z_i x_n$ .Equation (A2) must hold almost everywhere in $r_1, r_2 \ldots r_{n-1}$, implying an exact linear relationship. It can only be the case if the distribution is of less than full rank, violating the stated assumptions.

If $P = 0$, then the choice $\mathbf{z} = \mathbf{x}$ will plainly make $\mathbf{z}'H\mathbf{z} = 0$ without violating the full rank assumption.

☐

B Revelation experiments

As mentioned in section V, the task is to narrow down the efficient frontier in accordance with the risk preferences of a given investor. A potential starting point is the idea of the shadow risk free rate. Each point P along the OMD envelope has a different shadow risk free rate, obtained as $\rho(P) = \mu(P; R(\tilde{\mathbf{x}}(P)) = E[\tilde{R}U'_P(\tilde{R})]/E[U'_P(\tilde{R})]$; where the running mean function μ isdefined in section II. The idea is to identify the investor's representative P via the associated risk free rate. The steps might run as follows:

(a) For a trial P, chosen on the high side, find $\rho(P)$ as above. Also present to the client a plot of the density of portfolio returns , or alternatively the distribution function $F(\tilde{R})$, or its complement $1 - F(\tilde{R})$, in the form of a readily understandable schedule.

(b) Ask the client to imagine the opportunity to invest risk - free at the rate ρ.Then reduce ρ by just a little and ask the client if he or she would still want to hold some of the risk free asset at the new rate. If the answer is still yes, reduce P.

(c) Repeat until the investor is just willing to finally abandon the risk free asset and hold $\tilde{R} = R(P)$ alone. The corresponding value of P is that required, and the optimal portfolio is $\tilde{x}(P)$.

Less risk averse investors will stop earlier, at a higher ρ value, than the more risk averse, so their effective P is higher. The reader may check that this is so off a standard mean variance construction , in which context the actual or notional risk free rate parametises the efficient frontier. The shadow risk free rate is higher for less risk averse investors. As a test of consistency, one could alternatively present sucessively higher values of the risk free rate, noting the point at which the investor wishes to commence adding the risk free asset.

Very possibly one could design alternative revelation experiments. An objective in doing so should be to present the investor with a complete picture of the trial distribution of portfolio returns, allowing asymmetries to be taken into account, rather than just limiting information to the mean and variance at each stage.

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Footnotes

[1]As $X = \{\mathbf{x} : \mathbf{x}'\mathbf{1} = 1\}$, the first order conditions give $\frac{\partial V}{\partial x_j} = \lambda$, a Lagrange multiplier. As $\frac{dv}{dP} = \frac{\partial V}{\partial P} + \sum_j \frac{\partial V}{\partial x_j}\frac{\partial x_j}{\partial P}$ and $\sum \frac{\partial x_j}{\partial P} = 0$, it follows that $\frac{dV}{dP} = \frac{\partial V}{\partial P}$.

[2]Note that the pencil property may not necessarily hold in the presence of additional portfolio constraints. For instance, with additional nonnegativity constraints, the first order conditions reduce to $E[(r_i - R)U'(R)] = \mu_i$ where μ_i is a Kuhn Tucker multiplier such that $\mu_i x_i = 0$.So the τ_i values would not be zero at the optimum.

[3]See Appendix A for the precise definition of the Dirac delta function used