Investment, Uncertainty, and Liquidity*

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Abstract

We analyze the investment timing problem of a firm subject to a financing constraint. The threat of future funding shortfalls encourages the firm to accelerate investment beyond the level that is first-best optimal. Thus, our model highlights a new way by which costly external financing can distort investment behavior. Moreover, hedging is useful not only because it allows investment to proceed, but also because it allows investment to be delayed. These results can potentially help explain observed empirical relationships between investment and liquidity, investment and uncertainty, investment and hedging, and shareholder wealth and volatility.

1 Introduction

When there are informational asymmetries (e.g., Greenwald, Stiglitz and Weiss, 1984; Myers and Majluf, 1984), external financing can be costlier than internal financing. Consequently, investment projects that have positive net-present-value (NPV) under internal financing can have negative NPV if the firm has insufficient internal funds to finance all profitable investments. In this situation, the firm is subject to a financing constraint that results in investment being less than its first-best level — the underinvestment problem. Moreover, this effect may not be temporary; Minton and Schrand (1999) find that internal funding shortfalls lead to a permanent decrease in firm investment.

Theoretical models of the underinvestment problem typically assume a static investment environment where the standard NPV rule applies. However, more recent work has shown that the NPV rule may be sub-optimal in a dynamic environment. One of the more influential contributions in this area is the investment timing model of McDonald and Siegel (1986).[1] In that model, the firm has perpetual rights to a project and seeks to choose the investment date which provides the highest expected payoff. Because the project has an uncertain future value and an irreversible investment cost, the optimal policy is to invest only when the project's NPV exceeds a positive threshold reflecting the value of further delay. This contrasts with the conventional prescription that investment is justified whenever NPV is positive, thus emphasizing the importance of dynamic considerations when investment is irreversible and uncertain.

[1]For examples of similar models, see Bernanke (1983), Brennan and Schwartz (1985), Majd and Pindyck (1987), Triantis and Hodder (1990), Ingersoll and Ross (1992). Dixit and Pindyck (1994) provide an excellent overview of this literature.

The McDonald and Siegel (1986) model assumes that the investment decision can be made independently of the financing decision.[2] Although Mauer and Triantis (1994) extend the model to allow for the simultaneous determination of investment and the debt/equity mix, they do not consider the possibility that the firm may face a binding finance constraint. In this paper, we examine the implications of such a constraint for investment in a dynamic environment. Specifically, we consider the investment timing decision of a firm that must rely on the cash generated by its existing assets in order to finance any new investment. If, at any date, the firm's cash stock is less than the cost of a particular project, then investment in that project cannot occur at that date. Thus, we explicitly permit the possibility of funding shortfalls. Although total reliance on internal funds is an extreme case, it provides an interesting contrast with the standard, but equally extreme, approach where internal funds can be completely ignored.[3] Moreover, it approximates the situation faced by many firms, particularly small ones, and is consistent with the evidence of Minton and Schrand (1999) that firms forgo investment rather than access capital markets.[4]

The requirement that investment be financed internally restricts the states in which the firm can invest, so it lowers both the investment profitability threshold and the value of the project rights. Thus, our model highlights a new way by which costly external financing can distort investment behavior: the threat of a future funding shortfall reduces the value of a firm's timing options and leads to sub-optimal early investment. In other words, financing constraints can not only discourage investment, but also accelerate it. Although we are unaware of any work that explicitly compares the speed of the investment process across different types of firm, this result is consistent with the standard folklore that smaller and more marginal (and therefore more financially-constrained) firms are more aggressive about entering new markets or launching new products than bigger, safer and less financially constrained firms. This phenomenon is typically attributed to differences in risk attitudes (i.e., more caution on the part of unconstrained firms) or to differences in management and bureaucracy structure (i.e., slower decision-making by unconstrained firms), but our model suggests another explanation.

Our model also has implications for other aspects of investment policy. First, while an increase in the firm's current cash stock may encourage investment by lowering the cost

[2]The same is true of more complex models that allow for partial reversibility, expandability, and industry structure effects. See, for example, Dixit (1989), Abel et al. (1996), and Dixit and Pindyck (1994).

[3]Our results require only that the firm face potential financing constraints, so they also apply when it has access to (limited) external funds. However, as we discuss below, this would considerably complicate the analysis required to solve our model, so we maintain the simplifying assumption that investment can be financed only with cash.

[4]As Alsop (2001) notes, it may also accurately reflect the situation faced by venture capital firms following the 2000–2001 NASDAQ collapse.

of capital, it also lowers the risk of future funding shortfalls and thereby raises the profitability threshold required to justify immediate investment. Moreover, because the risk of future funding shortfalls is greatest for firms with low current liquidity, the effect on the threshold of an increase in the current cash stock is greatest for these firms. Thus, low-liquidity firms can have a lower investment-cashflow sensitivity than high-liquidity firms, consistent with the evidence of Cleary (1999) and Kaplan and Zingales (1997). Second, greater volatility in the firm's future cashflow distribution affects investment in an opposite manner to greater volatility in the project's future payoff distribution. Greater payoff volatility increases the value of investment delay and lowers current investment, but greater financing volatility raises the risk of future funding shortfalls and thereby lowers the value of waiting and increases current investment. As most feasible measures of uncertainty are likely to incorporate both types of volatility, our model therefore suggests one reason why empirical research finds little or no short-term relationship between investment and uncertainty.[5]

We also consider the optimal hedging policy of a firm subject to a financing constraint and show that this leads to an additional motivation for hedging. When investment must be funded from internal sources, any delay of investment exposes the firm to the risk that it may lose the ability to finance the project. Without hedging, the firm might have to rush into investment. But hedging mitigates the risk of funding shortfalls and allows the firm to avoid sub-optimal early investment. Thus, hedging adds value not only because it allows the firm to undertake profitable investment (as in Lessard, 1990, and Froot, Scharfstein and Stein, 1993), but also because it allows investment to be delayed. This suggests an explanation for the empirical finding that hedging firms have similar investment rates to non-hedging firms. By reducing cashflow volatility, hedging reduces the number of states in which investment is financially-constrained, but in so doing raises the threshold required to justify investment. The first effect increases investment, but the second effect decreases it.

In related work, Mello and Parsons (2000) consider a financially-constrained firm with stochastic input and output prices and derive the optimal operating and hedging policies. Like us, they emphasize the role of the financing constraint in restricting the firm's options. However, because they focus on the operating policy of a firm that has already invested, they do not consider the effects of the financing constraint on the initial decision to invest. Our focus on the investment timing decision therefore complements their analysis.

In the next section, we set out our model and compare the investment decision of the unconstrained firm with that of the financially-constrained firm. In Section 3, we explain how these effects can shed some light on various aspects of observed investment behavior. Section 4 examines the role of hedging and Section 5 contains some concluding remarks.

[5]Minton and Schrand (1999) report a negative long-run relationship between cashflow volatility and investment.

2 Investment under uncertainty

2.1 The unconstrained firm

In order to provide a benchmark for analyzing the effects of an investment financing constraint, we begin by briefly summarizing the investment timing model developed by McDonald and Siegel (1986) and simplified by Dixit and Pindyck (1994). In that model, a firm owns the rights to an investment project and has the option to invest in this project at any time. If the firm invests, it pays a fixed amount I and receives a project worth V. Project value follows the geometric Brownian motion process

$$dV = \mu V dt + \sigma V d\epsilon \tag{1}$$

where μ and σ are constant parameters and $d\epsilon$ is the increment of a Wiener process. In this situation, the firm invests if and only if V exceeds some fixed threshold $\hat{V}^u$, where we use the superscript 'u' to denote that this is the investment threshold for the unconstrained firm. Let $F^u(V) = F(V; \hat{V}^u)$ denote the value of the investment option when the current value of the project is V and the threshold is $\hat{V}^u$. Then the optimal investment policy consists of choosing the threshold $\hat{V}^u$ that maximizes F^u. Standard replication arguments imply that, prior to investment, F^u satisfies the differential equation

$$\frac{1}{2}\sigma^2 V^2 F^u_{VV} + (r - \delta) V F^u_V - r F^u = 0, \tag{2}$$

where subscripts denote partial derivatives, r is the riskless interest rate and δ is the opportunity cost of cashflows forgone due to waiting (henceforth the project's "dividend yield"). Given the boundary conditions

$$F^u(0) = 0, \quad F^u(\hat{V}^u) = \hat{V}^u - I,$$

equation (2) has the unique solution

$$F^u = (\hat{V}^u - I)\left(\frac{V}{\hat{V}^u}\right)^\beta \tag{3}$$

where

$$\beta = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2} > 1.$$

Maximizing (3) with respect to $\hat{V}^u$ yields the optimal investment threshold

$$\hat{V}^u = \frac{\beta I}{\beta - 1},$$

and investment option value

$$F^u = \left(\frac{I}{\beta - 1}\right)^{1-\beta}\left(\frac{V}{\beta}\right)^\beta.$$

It is straightforward to show that $\beta > 1$, so $\hat{V}^u > I$. That is, there are positive payoff ($V > I$) states in which the firm does not invest. In doing so, the firm retains the opportunity to receive potentially higher payoffs should project value rise (or avoid losses if project value falls). Similarly, this upside potential ensures $F^u \geq \max\{0, V - I\}$. However, these outcomes assume that project financing is guaranteed at all dates and thus ignore the possibility that waiting may eliminate the firm's ability to finance the project. To understand the implications of this for the investment timing decision, we next consider the situation where the firm's financing choices are restricted.

2.2 The constrained firm

We assume that the firm is restricted to financing the project with internal funds, i.e., there is no access to external capital markets.[6] Although the source of this constraint is immaterial for our purposes, it could arise for any of a number of reasons: informational asymmetries of the kind envisaged by Myers and Majluf (1984), irrationally low equity prices as in Baker et al. (2001), the types of agency problems described by Stulz (1990) or Myers (1977), or because the firm does not wish to reveal information to competitors about the project at the investment stage.

The firm begins with an initial cash balance X which, over time, is augmented in two ways. First, if the firm does not launch the project, X is invested in riskless securities. Second, the firm's existing physical assets generate operational cashflow. Thus, prior to investment in the project, the firm's cash stock follows the process

$$dX = rXdt + \nu dt + \phi d\zeta, \tag{4}$$

where ν and ϕ are constant parameters and $d\zeta$ is the increment of a Wiener process with $d\epsilon d\zeta = \rho dt$. Although it plays no formal part in our analysis, it may be helpful to think of (4) as describing a firm with a "lumpy" investment schedule. Additions or extensions to its existing stock of physical assets can take place only in indivisible units and, while the firm is waiting for sufficient funds to accumulate, the existing cash stock is placed in short-term securities. In the meantime, the firm's existing assets continue to augment (or deplete) the cash stock. As we wish to focus on the firm's investment strategy, and not on its financial distress policy, we assume that the firm loses the rights to the project if its cash stock becomes non-positive.

Note that equation (4) permits the interpretation of X as "available financing" rather than "cash stock" so long as increments to the external component of X are perfectly correlated with increments to the cash component. In this case, cash is a constant proportion of available financing and the first term on the right side of (4) is simply multiplied

[6] This approach is similar in spirit to the cash-in-advance models of individual investor demand. For discussions of that model, see Clower (1967) and Kohn (1981).

by that constant. In more general cases, however, the cash proportion is not constant and equation (4) cannot describe the evolution of available financing.

Investment is allowed if and only if $X \geq I$, so the level of internal funds places restrictions on the states in which the investment option can be exercised. This ensures that (i) the value of the investment option at any time prior to exercise depends on X as well as V and (ii) the investment threshold is not a constant as in the case of the unconstrained firm, but instead is a function of X.[7] Let $\hat{V}^c(X)$ denote this investment threshold function and $F^c(X,V) = F(X,V;\hat{V}^c)$ denote the value of the constrained investment option. Then the optimal investment policy consists of choosing the threshold function $\hat{V}^c(X)$ that maximizes F^c. In this case, F^c satisfies the differential equation (see the appendix for details)

$$\frac{1}{2}\sigma^2 V^2 F^c_{VV} + \frac{1}{2}\phi^2 F^c_{XX} + \rho\sigma\phi V F^c_{XV} + (r-\delta)VF^c_V + r(X+G)F^c_X - rF^c = 0, \quad (5)$$

where G is the market value of a claim to the future cashflow generated by the firm's existing physical assets. If $V \geq \hat{V}^c(X)$ and $X \geq I$, then $F^c = V - I$; otherwise it satisfies equation (5). In addition, the solution to (5) must satisfy (i) the same boundary conditions as the unconstrained firm and (ii) $F^c(0,V) = 0$.

The greater complexity of this model means that an analytical solution for F^c is unknown. However, it is clear that

$$\begin{aligned} \hat{V}^c(X) &\leq \hat{V}^u, \quad \forall X \geq I, \\ F^c(V,X) &\leq F^u(V). \end{aligned}$$

The basis for these differences between constrained and unconstrained firms is as follows. First, there are low X states in which the unconstrained firm would exercise the investment option, but the constrained firm has insufficient internal funds and so must continue to wait. Second, there are intermediate X states in which the unconstrained firm would choose to delay investment, but the benefits of doing so for the constrained firm are outweighed by the risks of losing the ability to finance the project. Thus, the additional investment constraints faced by the financially constrained firm are two-fold: in some states it cannot *begin* investment when it wishes to do so; in other states it cannot *delay* investment when it wishes to do so. The potential for these outcomes lowers the value of the project rights to the constrained firm. Moreover, the potential loss of financing for a currently-profitable project causes the constrained firm to adopt a lower threshold than

[7] Note we are implicitly assuming either that the project is in some way unique to the firm or that the rights are not tradeable. If the firm could freely trade the project rights, then the effect of the financing constraint is weakened because the firm could simply sell the rights at the optimal investment date if it had insufficient funds to invest itself. In this case, F^c would reflect the value of the project to other (potentially less-constrained) firms. In practice, the extent to which a firm could fully realize a project's value in this way is limited, so we ignore this complication.

Table 1: Baseline parameter values used in the numerical solution procedure

Parameter	Value
Project investment cost	$I = \$100$
Project value volatility	$\sigma = 0.2$
Project dividend yield	$\delta = 0.03$
Riskless interest rate	$r = 0.03$
Project value-firm cashflow correlation	$\rho = 0.5$
Cashflow volatility	$\phi = \$60$
Market value of existing physical assets	$G = \$100$

the unconstrained firm. Using numerical techniques, we next explore these differences in greater detail.

2.3 A numerical solution

We solve for the constrained firm's optimal investment timing policy using a numerical procedure based on finite difference methods, the details of which are provided in the appendix. Implementation requires that we specify values for the model parameters. Although we also consider the sensitivity of our results to alternative parameter values, we begin with the benchmark set of values appearing in Table 1. Most of the values appearing in Table 1 are similar to those used by other authors, e.g., Milne and Whalley (2000) and Mauer and Triantis (1994). The additional parameters are G, ρ and ϕ. Although the choice of G is necessarily arbitrary, setting it equal to \$100 means that firms with low current liquidity ($X < 100$) expect to receive a greater proportion of future increments to their cash stocks from the cashflow generated by their existing physical assets than from the interest return on their existing cash stocks.[8] Setting the correlation between X and V to 0.5 is consistent with the investment project having similar, but not identical, characteristics to the firm's existing assets. Finally, given $G = \$100$ and $r = 0.03$, $\phi = \$60$ is chosen to correspond with actual corporate data.[9]

Table 2 provides an initial indication of the effects of financing constraints on the

[8] This seems reasonable insofar as firms that are currently financially constrained are motivated to improve the efficiency of their existing assets in order to break free of the financing constraint. For firms with higher liquidity ($X \geq 100$), the primary expected contribution to their cash stocks is from the return on their existing cash. Again, this seems reasonable if, for example, firms that have accumulated high X have done so by skimping on additions to their stock of physical assets.

[9] Since rG must equal certainty-equivalent cashflow, the choice of $r = 0.03$ and $G = \$100$ yields certainty-equivalent cashflow of \$3. Assuming no systematic cashflow risk, the choice of $\phi = \$60$ implies a ratio of cashflow mean to cashflow standard deviation of 1/20, approximately the value found for US firms listed in the COMPUSTAT database between 1995 and 1999.

Table 2: Investment threshold and option values for unconstrained and constrained firms

Cash Stock	Investment Thresholds		Option Values	
X	$\hat{V}^u$	$\hat{V}^c$	F^u	F^c
100	220	141	8.06	4.33
150	220	180	8.06	5.35
200	220	190	8.06	5.99
250	220	200	8.06	6.43

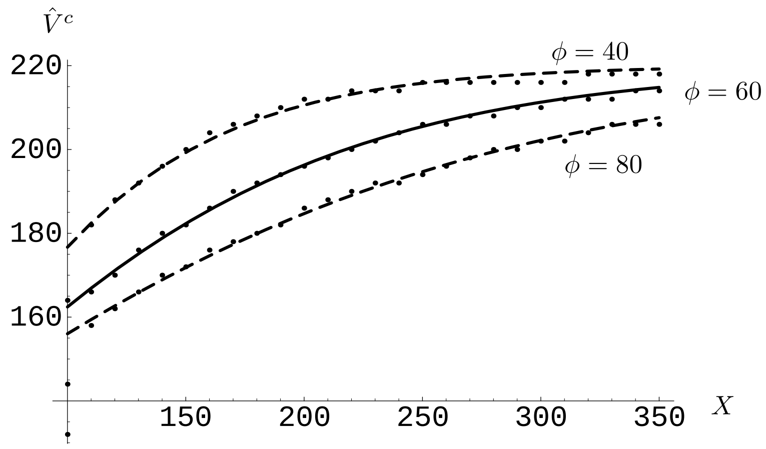
This table reports the investment thresholds ($\hat{V}^u$ and $\hat{V}^c$) and investment option values (F^u and F^c) for unconstrained and constrained firms respectively. X denotes the firm's current cash stock. Parameter values used in generating the threshold and option values are those given in Table 1. In addition, for calculating F^u and F^c, we assume initial project value V equals 100.

investment timing decision. For various values of initial cash stock X, we calculate the investment thresholds and option values for both unconstrained and constrained firms.[10] These calculations support our hypotheses that the financing constraint lowers both the investment threshold and the option value. For the unconstrained firm with parameters as given in Table 1, investment should be delayed until the current project value of $100 reaches $220; given this policy, the current value of the project rights is $8.06. However, delay for the constrained firm incurs the risk that the firm's cash stock will drop below $100, thereby making investment (temporarily, at least) impossible. This additional risk makes waiting less valuable, so the optimal investment policy for the constrained firm requires a lower threshold than for the unconstrained firm. This in turn lowers the value of the investment option. For example, when the constrained firm's current cash stock is just sufficient to cover the investment cost ($X = \$100$), the threshold is $141 and the opportunity cost of immediate investment ($\hat{V}^c - I$) is $41, some 66% less than for the unconstrained firm. Not surprisingly, this lowers the value of the investment option, in this case to $4.33, a level some 46% below its unconstrained value. Even when the firm's current cash stock is double that needed for investment, the investment option value for the constrained firm is 20% below its unconstrained counterpart. Overall, the existence of a financing constraint may cause the firm to sacrifice a significant proportion of a project's potential value.

A more general picture of the effects of the financing constraint appears in Figures 1 and 2. In Figure 1, we display the relationship between the investment threshold ($\hat{V}^c$) and the firm's initial cash stock (X). As X rises above I, the risk that the firm will

[10] For the purposes of calculating the option values, the initial project value V is set equal to the investment cost $I = \$100$, so the project has significant waiting value.

Figure 1: The constrained firm's investment threshold function



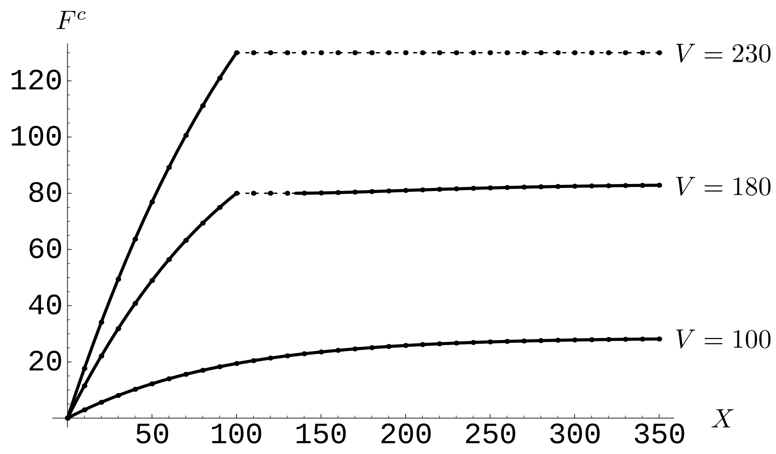
The value of the constrained firm's investment threshold is plotted for different values of initial cash stock (X) and cashflow volatility (ϕ). Parameter values are given in Table 1. For given ϕ, a rise in the initial cash stock decreases the risk that the firm will have insufficient cash to finance the project in the future, thereby increasing the value of waiting and raising the investment threshold. For given X, a rise in cashflow volatility increases the risk that the firm will have insufficient cash to finance the project in the future, thereby decreasing the value of waiting and lowering the investment threshold.

have insufficient cash to finance the project in the future falls, thereby increasing the incentive to wait (in order to learn more about project value) and raising the investment threshold. Initially, this effect is strong as increases in X from a low level significantly reduce the probability of future funding shortfalls.[11] Eventually however, the risk of such shortfalls becomes trivial, so further increases in X have little effect and the constrained firm's threshold converges on that of the unconstrained firm. Figure 1 also illustrates the influence of cashflow volatility ϕ. For each X, greater cashflow volatility increases the amount by which the constrained firm's threshold deviates from its unconstrained counterpart. The greater is ϕ, the greater the likelihood that adverse cashflow shocks will eliminate the firm's ability to finance the project when it wishes to invest. In response, the firm reduces its exposure to this risk by lowering the investment threshold. We can think of this as a formalized "bird-in-the-hand" strategy; a relatively small payoff received soon and with low risk is preferable to a potentially large payoff received later if there is significant risk of the latter payoff becoming zero due to a funding shortfall.[12]

[11]This occurs because higher X results in greater interest income, but has no effect on future cashflow volatility.

[12]One implication of Figure 1 is that investment hurdle rates rise with firm liquidity. Surveys by Summers (1987) and Poterba and Summers (1995) report hurdle rates well in excess of any reasonable cost of capital, a finding that Dixit and Pindyck (1994) suggest can be explained by investment timing considerations; investment incurs the opportunity cost of forgoing the option to wait, so the project

Figure 2: The constrained firm's investment option value

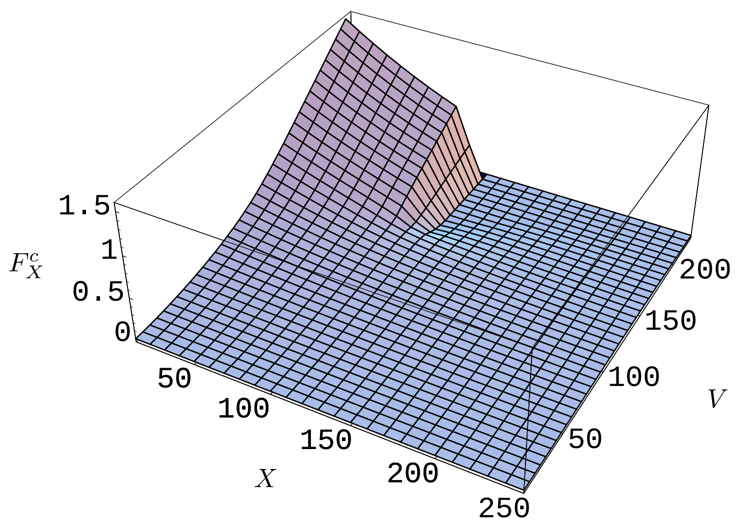


The value of the project rights for the constrained firm (F^c) is plotted for different values of initial cash stock (X) and project value (V). Parameter values are given in Table 1. For given V, a rise in X decreases the risk that the firm will have insufficient cash to finance the project in the future, thereby increasing the value of waiting and raising F^c. If V is low, the expected waiting time is high and so F^c increases monotonically with X. For intermediate V, an increase in X raises F^c for $X \leq I$. When $X = I$, the risk that the firm will have insufficient cash to finance the project in the future offsets the potential gains from waiting, so the project rights are exercised and additional increments in X have no effect on F^c until X is sufficiently high to reduce the risk of future funding shortfalls. If V is above the unconstrained threshold, then immediate investment is optimal, so F^c increases sharply with X for $X \leq I$, but is independent of X thereafter.

Figure 2 displays the relationship between the value of the investment option (F^c) and the firm's initial cash stock (X) for different project values (V). In general, higher X decreases the risk that the firm will have insufficient cash to finance the project in the future, thereby increasing the value of waiting and raising F^c. However, if V is low, the expected waiting time is long and so the funding shortfall risk is high even if X is currently well above the investment cost I. In this case, F^c increases monotonically with X until it converges on the unconstrained option value. By contrast, if V exceeds the unconstrained investment threshold (so that immediate investment is optimal), then F^c first rises sharply with X for values of X that are less than the investment cost I (because additional cash reduces the expected sub-optimal delay in investment), but is then independent of X beyond this point (because investment occurs and $F^c = V - I$). Finally, if V is greater than I, but less than the unconstrained investment threshold, the

hurdle rate contains a "premium" in recognition of this cost. If the Dixit and Pindyck hypothesis is correct, then our model implies that the cross-sectional variation in "excess" hurdle rates (i.e., the hurdle rate minus the firm's cost of capital) should reflect the cross-sectional variation in liquidity. That is, the excess hurdle rate should be an increasing function of firm liquidity.

Figure 3: The marginal effect of cash stock on the constrained firm's investment option value



The marginal value of initial cash stock for the constrained firm (F_X^c) is plotted for different values of initial cash stock (X) and project value (V). Parameter values are given in Table 1. When $V > I > X$, an additional dollar of cash can add more than \$1.50 to the value of project rights, thereby increasing firm value by \$2.50. When either $V < I$ or $X > I$, the effect on firm value is more modest, but frequently exceeds \$1.

relationship becomes more complex. Then, for $X < I$, F^c is strongly increasing in X because each additional dollar reduces the probability that the firm will face a funding shortfall when the optimal investment date arrives. For X equal to or slightly greater than I, the potential benefits of delaying investment are outweighed by the risk of subsequently losing the ability to finance the project, so the firm invests and additional increments to X have no effect on F^c (see the dashed line component of the $V = 180$ curve). However, for X sufficiently greater than I, the funding shortfall risk is small enough for investment delay to again become the optimal strategy. Additional increases in X then raise the value of the investment option until it converges on the unconstrained option value.

Our results suggest that the value to the firm of additional cash can be far greater than the face value of the cash. An illustration of the magnitude of this effect is provided in Figure 3 where we plot the marginal value of cash for the constrained firm (F_X^c). When $V > I > X$, an additional dollar of cash can add more than \$1.50 to the value of project rights, thereby increasing firm value by more than \$2.50. When either $V < I$ or $X > I$, the effect on firm value is more modest, but still exceeds \$1. The only exception to this occurs when V exceeds the unconstrained threshold and X is greater than I; in this case firm value changes only by the face value of the additional cash.

3 Some implications for investment issues

Since the work of Myers and Majluf (1984) and others, the underinvestment problem has received considerable attention. Underinvestment occurs when external financing is more costly than internal financing, thereby raising the cost of capital for firms that are unable to finance their investment plans from internal sources. This, in turn, lowers the profitability of all projects and causes some previously-profitable projects to become unprofitable. Our model suggests that dynamic considerations may introduce an additional distortion. When investment timing is flexible, costly external financing not only lowers project profitability, but also lowers the profitability hurdle required to justify investment. In other words, although a financing constraint discourages investment by causing projects to be less profitable, it also encourages investment by lowering the value of waiting for projects that can be delayed. Thus, in a dynamic framework, costly external financing can distort investment decisions by understating the value of waiting, thereby resulting in accelerated investment.

In the context of the investment-liquidity relationship, a recent debate has focused on whether or not the sensitivity of investment to firm liquidity is a useful measure of financing constraints. Beginning with an influential paper by Fazzari, Hubbard and Petersen (1988), the standard approach has been to divide a sample of firms into groups reflecting a priori rankings of likely financial constraints and then compare the investment-cashflow sensitivities of these different groups. Most studies find that the firms that a priori seem most likely to be financially constrained exhibit greater investment-cashflow sensitivity, thereby suggesting that the investment-cashflow sensitivity is indeed a useful measure of the severity of financing constraints. However, this approach has been criticized by Kaplan and Zingales (1997, 2000). Most tellingly, they find that within the sample of firms Fazzari, Hubbard and Petersen argue are most likely to be financially constrained, the firms with significant liquidity problems exhibit a *lower* investment-cashflow sensitivity than the firms that appear unlikely to have been financially constrained. Kaplan and Zingales stress that it is important to understand the source of this result and speculate that it may be due either to non-linearities in the external finance cost function or to currently unknown aversions to raising external finance.

Our model suggests another source. When the firm must finance investment from internal sources, greater cashflow not only increases current investment by lowering the cost of capital, but also decreases current investment by increasing the required payoff threshold. Thus, the sensitivity of investment to cashflow depends on the relative magnitudes of these two effects. Moreover, recall that (see Figure 1) the threshold-cash relationship is strongest for firms that are most financially constrained (low X) because these are the firms for whom delay poses the greatest risks. As a result, the threshold component of the investment-cashflow sensitivity is greatest for these firms. Since this component reduces investment, it follows that an increase in cash has a smaller positive effect on the investment of low-cash firms than it does on high-cash firms.[13] That is, the investment of highly-constrained firms is less sensitive to changes in cash than is the investment of firms facing weaker constraints, essentially the pattern observed by Kaplan and Zingales.[14] Our model therefore provides some support for the view that observed investment-cashflow sensitivities may indicate little about the severity of financing constraints.[15]

Our model can also shed some light on the relationship between investment and uncertainty. In the standard model of the unconstrained firm, all uncertainty eminates from the stochastic evolution of project value, so greater uncertainty increases the value of waiting and lowers investment. However, the empirical evidence for this relationship is inconclusive; Ghoshal and Loungani (1996, 2000) find the expected relationship between investment and price or profit uncertainty in industries with large numbers of small firms, but not for other industry structures; Caballero and Pindyck (1996) report a statistically significant relationship between investment and the variance of the marginal revenue product of capital, but one that is substantially smaller than predicted by the unconstrained firm model. Our model suggests one possible reason for these ambiguous results. In the presence of a financing constraint, both payoff and financing uncertainty exist and these have opposite effects on the investment threshold; payoff uncertainty raises the threshold while financing uncertainty lowers it. Thus, any attempt to empirically identify the relationship between uncertainty and investment will pick up offsetting uncertainty effects unless the exact nature of the uncertainty is carefully identified. For example, the uncertainty measures used in the above studies are based on historical estimates of volatility in some aspect of firm performance and thus seem likely to include aspects of both value and cashflow uncertainty. Consequently, it is unsurprising that the estimated relationship between investment and uncertainty is small or non-existent.

[13] To see this more formally, consider a firm with initial cash stock X_0 and project value V_0. Then it is straightforward to show that $\text{cov}(d(V - V^*), dX) = \phi(\rho\sigma V_0 - \phi V^*_X(X0))dt$. To the extent that $\rho > 0$, the first term inside the square brackets captures the cost of capital effect; the second term captures the threshold effect. Moreover, since $V^*_{XX} < 0$, $\text{cov}(d(V - V^*), dX)$ is an increasing function of X_0.

[14] Povel and Raith (2001) arrive at the same conclusion in a two-period model of investment when there is asymmetric information about the firm's revenue stream.

[15] We implicitly define a firm to be more financially constrained if it has fewer internal funds available for investment. A broader definition (and one more commonly used in the literature) of a more severe financing constraint is a greater wedge between the cost of external and internal funds. By this definition, all firms are equally constrained in our model (as none can access external funds), yet, as we have seen, they may have different investment-cashflow sensitivities. Thus, our conclusion — that observed investment-cashflow sensitivities may indicate little about the severity of financing constraints — also holds for this broader definition of financial constraints and serves as a complementary case to that of Alti (2001) who shows that the observed investment-cashflow relationships can occur in the absence of any market frictions.

The conflicting effects of payoff and financing uncertainty can also have implications for the interpretation of other empirical results. Shin and Stulz (2000) find that shareholder wealth is negatively related to equity price volatility (as a proxy for cashflow volatility), a result they attribute to financial distress costs. However, our model makes it clear that such a result is also consistent with real options models of investment decision-making; greater cashflow volatility reduces the value of the investment option (and therefore shareholder wealth) because it increases the likelihood of a future funding shortfall and therefore leads to sub-optimal investment timing.

4 The hedged firm

In the model of the previous section, the firm was provided with no means of mitigating or modifying its financing risk. In practice, firms can have various mechanisms available for hedging this risk. Indeed, Froot, Scharfstein and Stein (1993) stress that the primary motivation for hedging is to ensure that firms have the necessary financing for undertaking valuable investment projects. As we have seen, financing constraints not only cause firms to forgo investment, but also encourage premature investment. This suggests that another benefit of hedging is to ensure that firms have sufficient funding to confidently delay investment, thereby precluding the need for sub-optimal early launching due to concerns about future financing capabilities. In this section, we develop this and other hedging-related issues in more detail. To do so, we first derive and characterize the optimal hedge policy and then consider the effects of this policy on the investment threshold and option value.

We assume the firm can maintain a dynamic hedge. Specifically, at time t the firm holds h_t short positions in an asset or portfolio with price x_t whose returns are perfectly correlated with firm cashflow. The proceeds hx from this activity are held in a margin account earning interest at a rate $\bar{r} < r$. This arrangement ensures that the short positions can only be used to hedge operating cashflows (and not to raise cash) and that hedging is costly. In this case, the investment threshold $\hat{V}^h(X, h)$ depends on the hedging policy which in turn is reflected in the investment option value $F^h = F(X, V; \hat{V}^h)$. The optimal hedge policy is found by solving the Bellman equation (see the appendix for details)

$$\begin{aligned} rF^h \;=\; \sup_{h\geq 0} \Big\{ & \frac{1}{2}\sigma^2 V^2 F^h_{VV} + \frac{1}{2}(\phi - \sigma_x hx)^2 F^h_{XX} + \rho\sigma(\phi - \sigma_x hx) V F^h_{XV} \\ & + (r-\delta) V F^h_V + (r(X+G) - (r-\bar{r}))hx) F^h_X - (r-\bar{r})hx \Big\}, \end{aligned} \tag{6}$$

where σ_x is the standard deviation of returns on asset x. It follows that the value of the optimal hedge position is given by

$$h^* x = \frac{\phi}{\sigma_x} + \frac{\rho\sigma V F^h_{XV}}{\sigma_x F^h_{XX}} + \frac{(r-\bar{r})(1+F^h_X)}{\sigma_x^2 F^h_{XX}}. \tag{7}$$

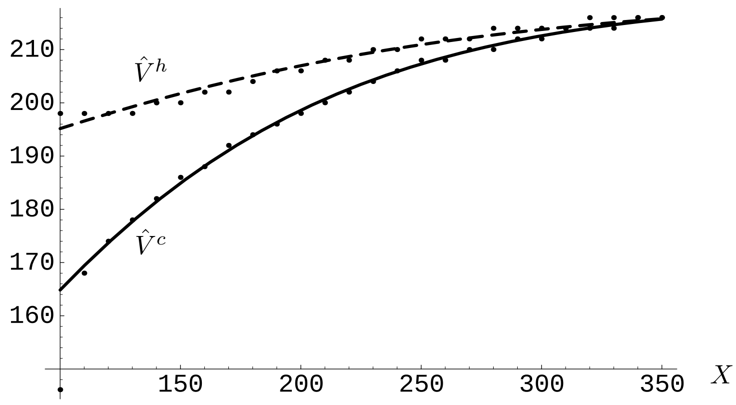
The first two terms on the right side of (7) represent the optimal demand for hedging in the absence of transactions costs; the last term captures the reduction in this optimal demand due to the forgone interest $(r - \bar{r})$ that hedging requires. To understand the meaning of the first two terms more fully, note first that any hedging policy typically reduces cashflow volatility by shifting cash from high-cash states to low-cash states. In equation (7), the first term gives the value of the complete hedge position, i.e., the hedge that completely eliminates random fluctuations in the firm's cashflow. However, shifting cash from a high-cash state to a low-cash state is counter-productive if the marginal value of cash is high in the former state, but low in the latter state. Consequently, the *optimal* hedging policy does not shift cash from all high-cash states to all low-cash states, but only from high-cash states in which the marginal value of cash is low to low-cash states in which the marginal value of cash is high.[16] Thus, if the correlation between V and X is positive ($\rho > 0$), then low-cash states also tend to be those in which cash is not very valuable, so the optimal hedge position is less than the complete hedge position. On the other hand, if $\rho < 0$, then low-cash states also tend to be those in which the value of cash is high, so the optimal hedge position is greater than the complete hedge position.[17]

The idea that the optimal hedging quantity is a decreasing function of the correlation between cashflow and project value has previously been discussed by Froot, Scharfstein and Stein (1993) in a model which shows that a firm should hedge in order to ensure that it is able to undertake profitable investments when they arise. In both our model and that of Froot, Scharfstein and Stein, positive correlation between X and V reduces the need for hedging because it implies that cashflow is low in states where the marginal value of cash is low. However, there is an important difference between their argument and ours. In the Froot, Scharfstein and Stein model, cash has value because it allows the firm to invest, so insufficient hedging may cause the firm to forgo investment. Thus, hedging adds value because it allows investment to occur. In our model, by contrast, cash has value because it allows the firm to retain the option to invest, so insufficient hedging may cause the firm to invest prematurely. Hedging adds value because it allows investment to be delayed.

[16] Mello and Parsons (2000) show that similar considerations apply to the situation where the firm is concerned with hedging the value of its operating options. It is straightforward to show, using a variant of the proof of their Proposition 2, that the hedging policy given by the first two terms in (7) minimizes $\mathrm{var}(F^h_X)$, the volatility of the marginal value of cash.

[17] This implicitly assumes that $F^h_{XV} > 0$. Although this is generally the case, the reverse sign holds in the region where V is just below the investment threshold. In that region, additional cash makes the value of the investment option less sensitive to changes in V, i.e., F_V declines in X. The reason is that extra cash gives the firm the flexibility to wait before investing, so the threshold increases and F_V falls below the value ($= 1$) it takes in the investment region. In this singular region, positive ρ means that low-cash states also tend to be those in which the value of cash is high, so the optimal hedge position is less than the complete hedge position.

Figure 4: The constrained firm's investment threshold function: hedged and unhedged



The value of the constrained firm's investment threshold function is plotted for (i) no hedging and (ii) optimal hedging. Parameter values are given in Table 1. The optimal hedging policy decreases the risk that the firm will have insufficient cash to finance the project in the future, thereby increasing the value of waiting and raising the investment threshold.

This principle can be seen in Figure 2 for the firm with $V = 175$. For X in the region of 100, the risk of losing the ability to finance the project discourages the firm from waiting and investment occurs immediately. Only when X rises sufficiently far above 100 does waiting again become optimal. In this situation, a suitable hedge increases the value of the investment option by reducing the number of states in which premature investment occurs.

The optimal hedge in our model can thus be greater or less than its counterpart in the Froot, Scharfstein and Stein (1993) model. On the one hand, allowing the firm to choose the timing of its investment reduces its need for hedging since it does not lose the project if funding is not available on a given date. On the other hand, the need to maximize the value of the investment option may increase the quantity of required hedging. For example, suppose X and V are perfectly positively correlated such that X exceeds 100 whenever V exceeds 100. Then the optimal Froot, Scharfstein and Stein hedge is zero. But for firms with an ongoing option to invest, the value of this option is positive even when $V < 100$ and, moreover, is enhanced by additional X. Consequently, firm value would be increased by a hedge that moved cash from states where X is more than sufficient to finance investment to states where X is low.

The effect of the optimal hedge on the investment threshold is displayed in Figure 4. For low values of X, the optimal hedge allows the firm to improve the timing of its investment and thus results in a higher threshold ($\hat{V}^h$). For higher values of X, the risk of future funding shortfalls is lower, so the need for hedging declines and the threshold approaches that of the non-hedging firm ($\hat{V}^c$). The difference in investment thresholds between hedging and non-hedging firms can potentially explain a puzzling aspect of the recent empirical literature on hedging. One implication of the Froot, Scharfstein and Stein (1993) basis for hedging is that hedging firms should, all else equal, invest more than non-hedging firms.[18] However, after adjusting for size differences, Allayannis and Mozumdar (2000) report little difference in the level of investment between the two groups of firms while Géczy, Minton and Schrand (1997) find that non-hedgers invest more than hedgers.[19] Our model suggests that these findings may be due to the ambiguous effect of hedging on the attractiveness of investment. On the one hand, hedging allows firms to undertake more investment by reducing the number of current states in which there is a funding shortfall. On the other hand, it also reduces the risk of future funding shortfalls and thus raises the investment threshold. Since these two effects work in opposite directions, it is not surprising that the data reveal little difference in investment between hedging and non-hedging firms.

5 Conclusion

When external finance is more costly than internal finance, firms may be forced to rely on internal funds to finance investment. Although this fundamental point has long been recognized, its implications for optimal investment timing have not previously been analyzed. In this paper, we consider the implications of imposing a "cash-in-advance"-type constraint on firm investment. Our principal conclusions are as follows:

1. A financing constraint lowers the value of waiting to invest because investment delay exposes the firm to the risk of future financing shortfalls.

2. Financing risk lowers the optimal investment threshold, thereby resulting in sub-optimal early investment. Thus, a financing constraint may cause the firm to sacrifice a significant proportion of a project's value not only by forcing it to forgo investment (as would be the case in static models), but also by forcing it to accelerate investment. This identifies a new way by which costly external financing can distort investment behavior.

3. Hedging protects the value of the firm's investment options and thus gives it the flexibility to delay investment. This identifies an additional rationale for hedging.

These basic results have implications for several unexplained empirical phenomena:

[18] Other theories (e.g., Stulz; 1999) also suggest this outcome. For example, if the costs of financial distress are material, then failing to hedge idiosyncratic risk may increase a firm's cost of capital and thereby depress investment.

[19] An obvious explanation for this is that non-hedgers in these samples have better investment opportunities than hedgers, but this is unsupported by the data as both studies report the latter as having higher Q and market-to-book values.

4. Greater cashflow permits more investment, but also raises the threshold required to justify investment. Since the latter effect is greatest for tightly-constrained firms, such firms may have lower investment-cashflow sensitivities than firms that are less constrained. This contrasts with the conventional view that high sensitivities indicate strong constraints, but is consistent with the evidence of Cleary (1999) and Kaplan and Zingales (1997).

5. Greater payoff uncertainty increases the threshold required to justify investment, but greater financing uncertainty decreases it. Since most empirical measures of uncertainty are likely to contain elements of both, their offsetting effects can help explain why the observed short-term relationship between investment and uncertainty is weak.

6. Cashflow hedging reduces the risk of future financing shortfalls, but in so doing increases the value of waiting to invest and thus raises the threshold required to justify investment. Thus, the effect of hedging on the level of corporate investment is ambiguous, consistent with existing empirical data.

Our analysis has focused on single stand-alone projects that have no effect on the financing constraints faced by other projects. An obvious extension of our work would consider the more general situation where the firm has a number of competing projects, each of which has different implications for the financing constraints faced by all others. For example, suppose a firm has two projects with the same positive NPV and cost I. If the firm is constrained (i.e., $I < X < 2I$), then launching one project now augments the future cash stock and thereby increases the likelihood of being able to subsequently launch the other. Thus, there is an additional incentive for the constrained firm to invest now. By contrast, if the payoffs to the two projects are negatively correlated, the unconstrained firm has an additional incentive to delay investment in order to see which turns out best. In this case, the differences that we have identified between constrained and unconstrained firms seem likely to be accentuated, but other cases may yield different outcomes, so further analysis of this complex problem has the potential to yield additional insights.

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A Appendix

A.1 Derivation of equation (5)

We assume that the risks inherent in V and X are spanned by the market of existing securities. Specifically, suppose that there are traded assets or portfolios with prices v and x that evolve according to

$$dv = \mu_v v dt + \sigma_v v d\epsilon, \quad \text{(A-1)}$$

$$dx = \mu_x x dt + \sigma_x x d\zeta. \quad \text{(A-2)}$$

Then a long position in the investment option can be combined with short positions of $\sigma V F_V/(\sigma_v v)$ units of asset v and $\phi F_X/(\sigma_x x)$ units of asset x to produce a total return dR over the time interval dt such that (for shorthand, we drop the 'c' superscript on F since it is obvious that we are referring only to the constrained firm)

$$dR = dF - \left(\frac{\sigma V F_V}{\sigma_v v}\right) dv - \left(\frac{\phi F_X}{\sigma_x x}\right) dx.$$

Using Itô's Lemma to obtain an expression for dF, substituting (A-1) and (A-2) for dv and dx respectively, and simplifying, we obtain

$$dR = \left(\frac{1}{2}\sigma^2 V^2 F_{VV} + \frac{1}{2}\phi^2 F_{XX} + \rho\sigma\phi V F_{XV} + \left(\mu - \frac{\mu_v \sigma}{\sigma_v}\right) V F_V + \left(rX + \nu - \frac{\mu_x \phi}{\sigma_x}\right) F_X\right) dt.$$

Since this return is riskless, the portfolio must earn the riskless rate of return. Therefore,

$$dR = r\left(F - \frac{\sigma V F_V}{\sigma_v} - \frac{\phi F_X}{\sigma_x}\right) dt.$$

Equating this to the above expression for dR means that F satisfies the differential equation

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^2 V^2 F_{VV} + \frac{1}{2}\phi^2 F_{XX} + \rho\sigma\phi V F_{XV} \\ &+ \left(\mu - \frac{\mu_v \sigma}{\sigma_v} + \frac{r\sigma}{\sigma_v}\right) V F_V + \left(rX + \nu - \frac{\mu_x \phi}{\sigma_x} + \frac{r\phi}{\sigma_x}\right) F_X - rF. \end{aligned} \quad \text{(A-3)}$$

Further simplification can most readily be obtained if we assume the expected returns μ_v and μ_x are given by some equilibrium model such as the CAPM. If the latter holds, then

$$\begin{aligned} \mu_x &= r + \rho_{xm}\sigma_x\lambda, \\ \mu_v &= r + \rho_{vm}\sigma_v\lambda, \end{aligned}$$

where ρ_{xm} $(= \rho_{Xm})$ and ρ_{vm} $(= \rho_{Vm})$ are the correlation coefficients of the market return with dx and dv respectively, and λ is the market price of risk. If $\delta \equiv \mu_v - \mu$ is the project's dividend yield, then

$$\mu + \delta = r + \rho_{vm}\sigma_v\lambda.$$

Hence, the (A-3) coefficient on VF_V, $\mu - (\mu_v\sigma/\sigma_v) + (r\sigma/\sigma_v)$, becomes

$$\mu - \rho_{vm}\sigma\lambda = r - \delta.$$

Now let G denote the market value of a claim to the future cashflow generated by the firm's existing physical assets. Clearly, from (4), G is independent of X and V, so $dG = 0$ over any time interval dt. Thus, the return on a long position in G consists only of the current cashflow $(\nu dt + \phi d\zeta)$. Hence, using (A-2), a long position in G combined with a short position in $\phi/(\sigma_x x)$ units of asset x yields a total return of

$$\nu dt + \phi d\zeta - \left(\frac{\phi}{\sigma_x x}\right) dx = \left(\nu - \frac{\phi\mu_x}{\sigma_x}\right) dt.$$

Since this return is riskless, we must have

$$\nu - \frac{\phi\mu_x}{\sigma_x} = r\left(G - \frac{\phi}{\sigma_x}\right),$$

which implies that the (A-3) coefficient on F_X, $rX + \nu - (\mu_x\phi/\sigma_x) + (r\phi/\sigma_x)$, is equal to $r(X + G)$. Making this substitution back into (A-3) yields (5).

A.2 Derivation of equation (6)

Over the time interval dt, the firm's initial cash stock pays interest equal to $rX_t dt$, the margin account pays interest equal to $\bar{r}h_t x_t dt$, and the firm's existing operations generate cashflow equal to $\nu dt + \phi d\zeta_t$. In addition, the firm must inject cash equal to $h_t dx_t$ into the margin account to maintain the required balance. Thus, the change in the firm's cash stock is

$$dX = (rX + \nu + (\bar{r} - \mu_x)hx)dt + (\phi - \sigma_x hx)d\zeta.$$

The firm now holds a portfolio consisting of a long position in the project rights, a short position in asset x, and an interest-bearing margin account. Over the time interval dt, the change in the value of this portfolio equals

$$dF^h - hdx + \bar{r}hxdt.$$

Using Itô's Lemma, this becomes

$$\begin{aligned}&\Big(\frac{1}{2}\sigma^2 V^2 F^h_{VV} + \frac{1}{2}(\phi - \sigma_x hx)^2 F^h_{XX} + \rho\sigma(\phi - \sigma_x hx)VF^h_{XV} + \mu V F^h_V \\ &\quad + (rX + \nu + (\bar{r} - \mu_x)hx)F^h_X + (\bar{r} - \mu_x)hx\Big)\, dt \\ &+ \big((\phi - \sigma_x hx)F^h_X - \sigma_x hx\big)\, d\zeta + \sigma V F^h_V d\epsilon,\end{aligned}$$

Applying standard replication arguments using the assets x and v yields the differential equation for F^h:

$$\begin{aligned}rF^h &= \frac{1}{2}\sigma^2 V^2 F^h_{VV} + \frac{1}{2}(\phi - \sigma_x hx)^2 F^h_{XX} + \rho\sigma(\phi - \sigma_x hx)VF^h_{XV} \\ &\quad + (r - \delta)VF^h_V + (r(X + G) - (r - \bar{r}))hx)F^h_X - (r - \bar{r})hx.\end{aligned}$$

It follows immediately that the optimal hedge policy satisfies (6)

A.3 Numerical solution procedure

The partial differential equation is solved on a grid with nodes $\{(X_k, V_j) : j = 1, \ldots, J,\ k = 1, \ldots, K\}$, where $X_k = k\,dX$ and $V_j = j\,dV$. At node (X_k, V_j), the resulting difference equation can be written in the form

$$\begin{aligned}0 &= a_j F_{j-1,k} + b_j F_{j,k} + c_j F_{j+1,k} + d_k F_{j,k-1} + e_k F_{j,k+1} \\ &\quad + f_j(F_{j+1,k+1} + F_{j-1,k-1} - F_{j-1,k+1} - F_{j+1,k-1}),\end{aligned}$$

where

$$\begin{aligned}a_j &= \frac{\sigma^2 V_j^2}{2dV^2} - \frac{(r - \delta)V_j}{2dV}, \\ b_j &= -\frac{\phi^2}{dX^2} - \frac{\sigma^2 V_j^2}{dV^2} - r, \\ c_j &= \frac{\sigma^2 V_j^2}{2dV^2} + \frac{(r - \delta)V_j}{2dV}, \\ d_k &= \frac{\phi^2}{2dX^2} - \frac{r(X_k + G)}{2dX}, \\ e_k &= \frac{\phi^2}{2dX^2} + \frac{r(X_k + G)}{2dX}, \\ f_j &= \frac{\rho\phi\sigma V_j}{4dX\,dV},\end{aligned}$$

and $F_{j,k} = F(X_k, V_j)$. This equation is defined whenever $2 \leq j \leq J-1$ and $2 \leq k \leq K-1$. We extend it to the edges of the grid using four boundary conditions. Two are given in the statement of the problem: $F(X, 0) = F(0, V) = 0$. We therefore define $F_{0,k} = F_{j,0} = 0$ for all j and k. Motivated by the observation that

$$F(X, V + dV) = 2F(X, V) - F(X, V - dV) + O(dV^2),$$

we define

$$F_{J+1,k} = 2F_{J,k} - F_{J-1,k}$$

for all k. Finally, we suppose that K is sufficiently large that $F_{j,K} = F(V_j; \hat{V}^u)$. That is, we use the value of the project owned by an unconstrained firm along the $X = X_K$ boundary.

We start by setting $F_{j,k} = 0$ if $X_k < I$ or $V_j < I$, and set $F_{j,k} = V_j - I$ at all other nodes, and then solve the system using the method of Successive Over Relaxation. During each iteration of this method, we solve the difference equation at each node (X_k, V_j) in turn, replacing the calculated value of $F_{j,k}$ with $V_j - I$ at any node for which $X_k \geq I$ and $F_{j,k} < V_j - I$. We stop iterating when the largest change in any $F_{j,k}$, measured relative to its value at the end of the preceding iteration, is less than $I/10000$.