# Reflection and transmission of compressional waves by a stratification with discontinuities in density and/or sound speed

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Formulas are derived for the reflection and transmission amplitudes due to a stratification with discontinuities in density $\rho$ or sound speed $c$, or both. The formulas are exact in the case of a uniform layer ($\rho$ and $c$ constant within the stratification), and accurate down to surprisingly low frequencies for nonuniform layers. At very low frequencies these formulas ultimately fail, but there a long-wave theory is available. An additional limitation on the applicability of the theory given here is that it fails at classical turning points: Thus total reflection and tunneling are excluded. Comparison with an exactly solvable stratification model is made, from zero frequency up.

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## INTRODUCTION

In a recent paper,[1] a variety of results were derived for the reflection and transmission of acoustic compressional waves by an arbitrary stratification, including high- and low-frequency limiting forms. A particular application of the high-frequency formulas was to a stratification that is smooth, except at a finite number of planes where there are discontinuities in the *derivatives* of the density $\rho$ or the sound speed $c$, or both. The purpose of this article is to extend these results to include the possibility of discontinuities in the *values* of $\rho$ and $c$ at the boundaries of the stratification, as well as in their derivatives. Such discontinuities are commonly found,[2] and thus a theory of reflection from such stratifications is desirable. The discontinuities dominate the reflection process, and a perturbation theory or the Rayleigh approximation (Sec. III of Ref. 1, and Ref. 3) do not provide an adequate starting point. An alternative formulation is given here, which leads to results that are exact in the high-frequency limit, and also exact for the uniform layer at all frequencies. At low frequencies the formulas derived here fail (except for the uniform layer), but there the limiting forms derived in Sec. II of Ref. 1 can be used, as we shall see.

The problem being discussed is shown schematically in Fig. 1. A plane acoustic pressure wave is incident from medium $a$ onto an arbitrary stratification, extending from depth $z = a$ down to depth $z = b$. A representative profile for $c$ is shown on the right. The sound speed $c(z)$ and the density $\rho(z)$ will, in general, have discontinuities in value as well as in derivative at the boundaries.

## I. FORMULAS FOR REFLECTION AND TRANSMISSION AMPLITUDES

For the geometry of Fig. 1, with a plane wave of angular frequency $\omega$ propagating in the $zx$ plane, the acoustic pressure $p$ has the form $p(z,x,t) = \exp i(Kx - \omega t)P(Z)$, where $K$ is the $x$ component of the wave vector, and is a constant of the motion [$K = (\omega/c_a)\sin\theta_a = (\omega/c_b)\sin\theta_b$, where $\theta_a$ and $\theta_b$ are angles of incidence and refraction]. The differential equation for $P(z)$ is[4-7]

$$\rho\frac{d}{dz}\left(\rho^{-1}\frac{dP}{dz}\right) + q^2P = 0,\quad q^2(z) = \frac{\omega^2}{c^2(z)} - K^2. \qquad (1)$$

In media $a$ and $b$, the normal component of the wave vector $q(z)$ takes the values

$$q_a = (\omega/c_a)\cos\theta_a,$$
$$q_b = (\omega/c_b)\cos\theta_b = (\omega/c_a)(c_a^2 - c_b^2\sin^2\theta_a)^{1/2}.$$

The reflection and transmission amplitudes $r$ and $t$ are defined by

$$P(z) = \begin{cases} e^{iq_az} + re^{-iq_az} & (z<a),\\ uF(z) + vG(z) & (a\leqslant z\leqslant b),\\ te^{iq_bz} & (z>b).\end{cases} \qquad (2)$$

Here, $F$ and $G$ are two independent solutions of (1) within the stratification, and $u$ and $v$ are constant coefficients. As shown in Ref. 1, if $F$ and $G$ are known, then $r$ and $t$ can be found from

$$r = e^{2i\alpha}\frac{Q_aQ_b(F,G) + iQ_a(F,\overline{G}) + iQ_b(\overline{F},G) - (F,\overline{G})}{Q_aQ_b(F,G) + iQ_a(F,\overline{G}) - iQ_b(\overline{F},G) + (F,\overline{G})}, \qquad (3)$$

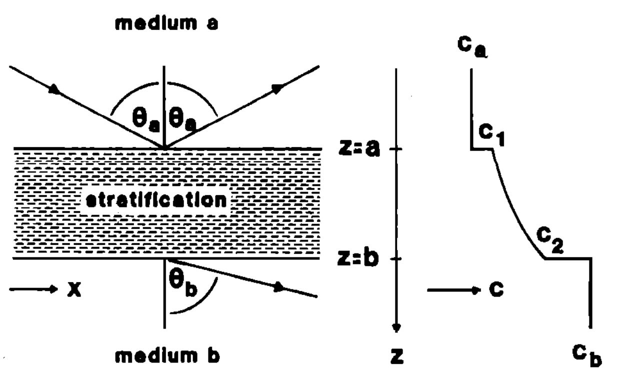


FIG. 1. Schematics of reflection and transmission by a nonuniform stratification with discontinuities in density $\rho$ and sound speed $c$ at its boundaries. Only the sound-speed profile is shown.

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$$t = e^{i(\alpha-\beta)} \frac{2iQ_a(F_2\overline{G}_2 - \overline{F}_2 G_2)}{Q_a Q_b (F,G) + iQ_a(F,\overline{G}) - iQ_b(\overline{F},G) + (\overline{F},\overline{G})}. \quad (4)$$

In (3) and (4) and in the remainder of this article, the following shorthand is used: $\alpha = q_a a$, $\beta = q_b b$, $Q_a = q_a/\rho_a$, $Q_b = q_b/\rho_b$,

$$(F,G) = F_1 G_2 - G_1 F_2, \quad (F,\overline{G}) = F_1 \overline{G}_2 - G_1 \overline{F}_2, \quad \text{etc.}, \quad (5)$$

where $F_1 = F(a+)$, $F_2 = F(b-)$; $\overline{F}_1$ stands for the derivative of $F$ at $z = a+$, divided by the value of $\rho$ just inside the stratification, namely $\rho_1$, $\overline{F}_2 = F'(b-)/\rho_2$, with similar notation for $G$.

In special cases one can find analytic solutions of (1), and then $r$ and $t$ can be found exactly. We are interested in getting approximate $r$ and $t$ for any discontinuous stratification of the type shown in Fig. 1. To this end we approximate $F$ and $G$ by the Liouville–Green waveforms (see Ref. 7, Sec. 6-2 for their properties, and a brief historical note that explains the name given to them, in contrast to the common WKB or JWKB designation):

$$F \approx (Q_1/Q)^{1/2} e^{i\phi}, \quad G \approx (Q_2/Q)^{1/2} e^{-i\phi}, \quad (6)$$

where $Q(z) = q(z)/\rho(z)$, $Q_1 = q_1/\rho_1$, $Q_2 = q_2/\rho_2$, and the phase integral $\phi(z)$ gives the accumulated phase at $z$:

$$\phi(z) = \int^z d\zeta\, q(\zeta). \quad (7)$$

The resulting approximate values of $(F,G)$ to $(\overline{F},\overline{G})$ are then

$$\begin{aligned}
&(F,G) \approx -2i \sin \Delta\phi, \\
&(F,\overline{G}) \approx iQ_2(-2\cos\Delta\phi + \gamma_2 \sin\Delta\phi), \\
&(\overline{F},G) \approx iQ_1(2\cos\Delta\phi + \gamma_1 \sin\Delta\phi), \\
&(\overline{F},\overline{G}) \approx iQ_1 Q_2[-2\sin\Delta\phi + (\gamma_1 - \gamma_2)\cos\Delta\phi \\
&\qquad - \tfrac{1}{2}\gamma_1\gamma_2 \sin\Delta\phi].
\end{aligned} \quad (8)$$

Here, $\Delta\phi$ is the phase increment on going through the stratification from $a$ to $b$:

$$\Delta\phi = \int_a^b dz\, q(z) = \frac{\omega}{c_a}\int_a^b dz \left(\frac{c_a^2}{c^2(z)} - \sin^2\theta_a\right)^{1/2}. \quad (9)$$

Also appearing in (8) are the (internal) boundary values $\gamma_1$ and $\gamma_2$ of the dimensionless function

$$\gamma(z) = \frac{dQ/dz}{qQ} = q^{-1}\left(\frac{1}{q}\frac{dq}{dz} - \frac{1}{\rho}\frac{d\rho}{dz}\right), \quad (10)$$

where $Q \equiv q/\rho$. This function, and its derivative divided by $q$, should be small throughout the stratification if the Liouville–Green functions (6) are to be good approximations to the exact solutions of (1), since from (6) we see that $F$ and $G$ satisfy the equation

$$\rho \frac{d}{dz}\left(\rho^{-1}\frac{dF}{dz}\right) + q^2\left(1 + \frac{1}{2}\frac{d\gamma}{d\phi} + \frac{1}{4}\gamma^2\right)F = 0. \quad (11)$$

Thus the approximations fail at low frequencies ($\gamma$ is proportional to $\omega^{-1}$) and also whenever $q$ is small, as happens at grazing incidence, and at classical turning points (zeros of $q$). From (1) we see that $q^2(z)$ stays positive, and classical turning points will not occur, if $c(z) < c_a/\sin\theta_a$. This inequality covers both total reflection, which occurs for $\sin\theta_a > c_a/c_b$, and the possibility of "tunneling" through a region of negative $q^2$ but with $c_b < c_a/\sin\theta_a$.

From (8), we find the reflection and transmission amplitudes on substituting into (3) and (4). We expand these in powers of $\gamma$:

$$r = r_0 + r_1 + \cdots, \quad t = t_0 + t_1 + \cdots. \quad (12)$$

The zeroth-order amplitudes are, with $c$ and $s$ short for $\cos\Delta\phi$ and $\sin\Delta\phi$:

$$r_0 = e^{2i\alpha}\frac{(Q_a Q_2 - Q_b Q_1)c - i(Q_a Q_b - Q_1 Q_2)s}{(Q_a Q_2 + Q_b Q_1)c - i(Q_a Q_b + Q_1 Q_2)s}, \quad (13)$$

$$t_0 = e^{i(\alpha-\beta)}\frac{2Q_a(Q_1 Q_2)^{1/2}}{(Q_a Q_2 + Q_b Q_1)c - i(Q_a Q_b + Q_1 Q_2)s}. \quad (14)$$

When $Q_1 = Q_2 = Q$, these reduce to the uniform layer values (given in different form in Refs. 4 and 5)

$$r_u = e^{2i\alpha}\frac{Q(Q_a - Q_b)c - i(Q_a Q_b - Q^2)s}{Q(Q_a + Q_b)c - i(Q_a Q_b + Q^2)s}, \quad (15)$$

$$t_u = e^{i(\alpha-\beta)}\frac{2Q_a Q}{Q(Q_a + Q_b)c - i(Q_a Q_b + Q^2)s}. \quad (16)$$

Note that a nonuniform layer could have $Q_1 = Q_2$; to zeroth order in $\gamma$, the approximation used here will give the same reflection and transmission amplitudes as a uniform layer, but a correction appears in the first-order terms.

The contributions of first order in $\gamma$ to the reflection and transmission amplitudes are

$$r_1 = e^{2i\alpha}\frac{iQ_a Q_1[Q_2^2(\gamma_2 - \gamma_1)c^2 + (Q_b^2\gamma_1 + Q_2^2\gamma_2)s^2 + 2iQ_b Q_2\gamma_1 cs]}{[(Q_a Q_2 + Q_b Q_1)c - i(Q_a Q_b + Q_1 Q_2)s]^2}, \quad (17)$$

$$t_1 = e^{i(\alpha-\beta)}\frac{Q_a(Q_1 Q_2)^{1/2}[(Q_a Q_2\gamma_2 - Q_b Q_1\gamma_1)s - iQ_1 Q_2(\gamma_1 - \gamma_2)c]}{[(Q_a Q_2 + Q_b Q_1)c - i(Q_a Q_b + Q_1 Q_2)s]^2}. \quad (18)$$

When there is no discontinuity in $Q$ at either boundary ($Q_1 = Q_a$ and $Q_2 = Q_b$), $r_0$ is zero and $t_0$ reduces to the perfect transmission value $e^{i(\alpha-\beta)}(Q_a/Q_b)^{1/2}e^{i\Delta\phi}$, while $r_1$ takes the value

$$e^{2i\alpha} i(\gamma_2 e^{2i\Delta\phi} - \gamma_1)/4,$$

which is equivalent to Eq. (51) of Ref. 1.

The above theory is based on the assumption that $\gamma$ and its derivative $d\gamma/d\phi = q^{-1}\, d\gamma/dz$ are both small. The approximations thus fail at low frequencies (except for the uniform layer, for which $\gamma$ is identically zero). There we have the long-wavelength expansions derived in Sec. II of Ref. 1.

To second order in the interface thickness (order being the power of a dimensionless quantity like $\omega\Delta z/c_a$), the reflectivity is given by four integrals, namely:

$$
\begin{aligned}
I_1 &= \int_a^b dz\,\rho(z), \quad J_1 = \int_a^b dz\,\frac{q^2(z)}{\rho(z)},\\
I_2 &= \int_a^b dz\,\rho(z) \int_a^z d\zeta\,\frac{q^2(\zeta)}{\rho(\zeta)}, \qquad (19)\\
J_2 &= \int_a^b dz\,\frac{q^2(z)}{\zeta(z)} \int_a^z d\zeta\,\rho(\zeta).
\end{aligned}
$$

When there is absorption, there is a first-order correction to the zero-thickness reflectivity

$$R_{\text{step}} = \left[\frac{(Q_a - Q_b)}{(Q_a + Q_b)}\right]^2. \qquad (20)$$

[The subscript step indicates a sudden (step) transition from medium $a$ to medium $b$.] When there is no absorption, the correction to $R_{\text{step}}$ is second order in the thickness of the stratification,[1]

$$
\begin{aligned}
R &= R_{\text{step}} + [4Q_aQ_b/(Q_a+Q_b)^4]\\
&\quad \times [(Q_aQ_bI_1)^2 + J_1^2 - 2Q_a^2J_2 - 2Q_b^2I_2] + \cdots . \qquad (21)
\end{aligned}
$$

The high- and low-frequency expressions will be compared with exact reflectivities for a solvable model profile in the next section.

## II. EXACT AND APPROXIMATE REFLECTIVITIES FOR THE EXP–EXP STRATIFICATION

In Sec. IV of Ref. 1, analytic solutions for $F$ and $G$ were given for two model stratifications. We will use the one in which both $\rho$ and $c$ vary exponentially with depth to compare the approximate expressions of the previous section with exact results. The exponential variation of $\rho$ and $c$ is defined in terms of two lengths:

$$\rho(z) = \rho_1 e^{(z-a)/l}, \quad l = \Delta z/\log(\rho_2/\rho_1), \qquad (22)$$

$$c(z) = c_1 e^{(z-a)/L}, \quad L = \Delta z/\log(c_2/c_1). \qquad (23)$$

For this stratification, the integrals defined in (19) take the values

$$
\begin{aligned}
I_1 &= (\rho_2 - \rho_1)l, \quad J_1 = \frac{\omega^2}{1/l + 2/L}\left(\frac{1}{\rho_1c_1^2} - \frac{1}{\rho_2c_2^2}\right)\\
&\qquad - lK^2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right),\\
I_2 &= \frac{\omega^2}{1/l + 2/L}\left[\frac{l}{c_1^2}\left(\frac{\rho_2}{\rho_1} - 1\right) - \frac{L}{2}\left(\frac{1}{c_1^2} - \frac{1}{c_2^2}\right)\right]\\
&\qquad - lK^2\left[l\left(\frac{\rho_2}{\rho_1} - 1\right) - \Delta z\right], \qquad (24)\\
J_2 &= \frac{lL}{2}\omega^2\left(\frac{1}{c_1^2} - \frac{1}{c_2^2}\right) - lK^2\Delta z\\
&\qquad - \frac{\omega^2 l}{1/l + 2/L}\left(\frac{1}{c_1^2} - \frac{\rho_1}{\rho_2c_2^2}\right) + K^2l^2\left(1 - \frac{\rho_1}{\rho_2}\right).
\end{aligned}
$$

[The reader is reminded that $K$ is the component of wave vector along the interface: $K = (\omega/c_a)\sin\theta_a$.]

For the high-frequency approximations, we need $\gamma_1$, $\gamma_2$, and $\Delta\phi$. From the defining relation (10) for $\gamma$, and using $Q = q/\rho$ and $q^2(z) = \omega^2/c^2(z) - K^2$, we find

$$\gamma(z) = -q^{-1}\left[\left(\frac{\omega}{cq}\right)^2 \frac{1}{c}\frac{dc}{dz} + \frac{1}{\rho}\frac{d\rho}{dz}\right]. \qquad (25)$$

Into this general expression, we insert the exp–exp profile values

$$\frac{1}{c}\frac{dc}{dz} = \frac{1}{L}, \quad \frac{1}{\rho}\frac{d\rho}{dz} = \frac{1}{l}, \qquad (26)$$

and then obtain $\gamma_1$ and $\gamma_2$ by substituting the values $q_1$ and $q_2$ for $q$ and $c_1$ and $c_2$ for $c$. The phase increment $\Delta\phi$ across the stratification, assuming no absorption and angle of incidence less than arcsin $(c_a/c_{\max})$ so that $q$ remains real, can be found analytically[1] for the exp–exp profile:

$$\Delta\phi = L\{K[\text{atn}(q_2/K) - \text{atn}(q_1/K)] + q_1 - q_2\}. \quad (27)$$

With these results, we can use the high-frequency formulas (13) and (17) for $r_0$ and $r_1$, and calculate the reflectivity

$$R_{\text{HF}} = |r_0 + r_1|^2. \qquad (28)$$

Figure 2 shows the reflectivity from a model exp–exp stratification, with acoustic parameters chosen to correspond to the Tufts abyssal plain, as presented by Chapman.[8] The exact reflectivity is obtained from the results of Sec. IV of Ref. 1, the low-frequency curve from (21) and (24), and the high-frequency curve from (13), (17), (25), (27), and (28).

We see that the low-frequency approximation is good up to about $\omega\Delta z/c_a = 1$ ($\lambda_a \geqslant 6\Delta z$), while the high-frequency results are good from about $\omega\Delta z/c_a = 2$ ($\lambda_a \leqslant 3\Delta z$). In the intermediate region, the errors can be 20% or more, and it may be necessary to use numerical methods[6,9–11] to obtain the reflectivity.

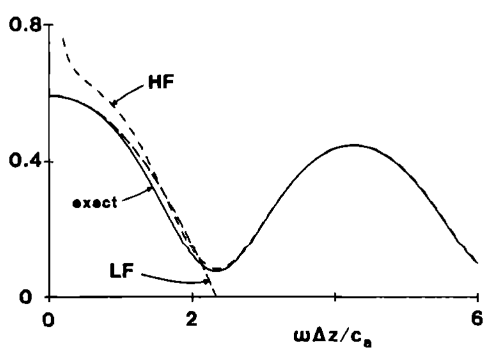


FIG. 2. Normal incidence reflectivities for an exp–exp stratification, as a function of frequency. Note that the dimensionless parameter $\omega\Delta z/c_a$ is $2\pi$ times the thickness of the stratification divided by the wavelength in medium $a$. The solid curve is the exact reflectivity, the dashed curves are the low- and high-frequency approximations, as indicated. The parameters used were: $\rho_a = 1$, $\rho_1 = 1.5$, $\rho_2 = 1.7$, $\rho_b = 2.2$ (g cm$^{-3}$), and $c_a = 1.5$, $c_1 = 1.7$, $c_2 = 2.3$, $c_b = 5.2$ (km s$^{-1}$).

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