Matrix methods for the calculation of reflection amplitudes

John Lekner and Michael C. Dorf

Department of Physics, Victoria University of Wellington, Private Bag, Wellington, New Zealand

Received December 18, 1986; accepted June 2, 1987

Improved numerical techniques for the calculation of the reflection of electromagnetic waves by an arbitrary stratified inhomogeneity are derived and assessed. The inhomogeneity is sectioned into thin layers, in each of which the dielectric function is taken to vary linearly, or cubically, with depth. The matrices representing each layer are calculated to third order in layer thickness. Some general properties of the matrices used in these calculations are also presented.

1. INTRODUCTION

We present improved matrix methods for the numerical evaluation of reflection amplitudes for s and p polarizations for an arbitrary planar stratified transition between two uniform media. The reflection amplitudes r_s and r_p give the reflectances $R_s = |r_s|^2$ and $R_p = |r_p|^2$ and the ellipsometric ratio r_p/r_s. Matrix methods are reviewed and extended in Chaps. 12 and 13 of a recent monograph (Ref. 1, hereafter referred to as TR). All matrix methods approximate the stratified inhomogeneous region by a set of N uniform or nonuniform layers; the reflection properties are then obtained from the product of N two-by-two *layer* matrices or $N + 1$ two-by-two *boundary* matrices. In the conventional approach, matrices relate the mutually perpendicular components of the electric and magnetic fields from layer to neighboring layer.[2] Here we use layer matrices introduced in TR, which relate the fields and their derivatives. In the absence of absorption, the matrix elements are all real. The standard approach leads to imaginary off-diagonal matrix elements, complicating the process of taking matrix products. Figure 1 illustrates how an inhomogeneous transition between uniform media of dielectric constants ϵ_a and ϵ_b is approximated by a set of layers in each of which the dielectric function varies linearly with depth.

For the s polarization, $\mathbf{E} = \{0, \exp[i(Kx - \omega t)]E(z), 0\}$, where K is the x component of the wave vector and is a constant of the motion. At z_n and z_{n+1}, $E(z)$ takes the values E_n and E_{n+1}, its derivative $D = \mathrm{d}E/\mathrm{d}z$ takes the values D_n and D_{n+1}, and these two pairs are linked by the matrix $M_n = \{m_{ij}\}$:

$$\begin{bmatrix} E_{n+1} \\ D_{n+1} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} E_n \\ D_n \end{bmatrix}. \tag{1}$$

If the uniform media on either side of the inhomogeneity are labeled a and b, and the wave is incident from medium a, $E(z)$ has the limiting forms

$$E = \exp(iq_a z) + r_s \exp(-iq_a z) \qquad (z \le z_1),$$
$$E = t_s \exp(iq_b z) \qquad (z \ge z_{N+1}). \tag{2}$$

Here q_a and q_b are the z components of the wave vector in media a and b. If s_{ij} are the matrix elements of $M_N M_{N-1} \ldots M_n \ldots M_2 M_1$, the s reflection amplitude is given by

$$r_s = \exp(2iq_a z_1) \frac{q_a q_b s_{12} + s_{21} + iq_a s_{22} - iq_b s_{11}}{q_a q_b s_{12} - s_{21} + iq_a s_{22} + iq_b s_{11}}. \tag{3}$$

For the p polarization, $\mathbf{B} = \{0, \exp[i(Kx - \omega t)]B(z), 0\}$, and the matrix M_n relates the values of B and $C = \mathrm{d}B/\epsilon \mathrm{d}z$ at z_n and z_{n+1}; $\epsilon(z)$ is the dielectric function. If now p_{ij} are the elements of $M_N \ldots M_n \ldots M_1$, the p reflection amplitude is given by

$$-r_p = \exp(2iq_a z_1) \frac{Q_a Q_b p_{12} + p_{21} + iQ_a p_{22} - iQ_b p_{11}}{Q_a Q_a p_{12} - p_{21} + iQ_a p_{22} + iQ_b p_{11}}, \tag{4}$$

where $Q_a = q_a/\epsilon_a$ and $Q_b = q_b/\epsilon_b$. (With the definition of r_p used in TR and here, $r_p = r_s$ at normal incidence, when there is no physical distinction between the two polarizations.)

The simplest approximation replaces $\epsilon(z)$ by a constant value, ϵ_n, within $z_n < z < z_{n+1}$. The matrices M_n then become, for the s and p polarizations, and with $\delta_n = q_n(z_{n+1} - z_n) \equiv q_n \delta z_n$,

$$\begin{bmatrix} \cos\delta_n & q_n^{-1}\sin\delta_n \\ -q_n \sin\delta_n & \cos\delta_n \end{bmatrix}, \qquad \begin{bmatrix} \cos\delta_n & Q_n^{-1}\sin\delta_n \\ -Q_n \sin\delta_n & \cos\delta_n \end{bmatrix}. \tag{5}$$

The numerical method based on these matrices taken to first order in δz_n is equivalent to the Euler method of solving the differential equations (TR, Sec. 13-1). Natural improvements are to go to higher order in δz_n and to allow a variable $\epsilon(z)$ within a layer. Law and Beaglehole[3] took a linear variation in $\epsilon(z)$,

$$\epsilon(z) = \epsilon_n + (z - z_n)\delta\epsilon_n/\delta z_n \tag{6}$$

($\delta\epsilon_n = \epsilon_{n+1} - \epsilon_n$), and the corresponding matrix to first order in δz_n. In TR the linear approximation was taken to second order in δz_n. Here we derive expressions for matrices for linear and cubic variations in $\epsilon(z)$ taken up to third order in δz_n. But first we consider some general properties of such matrices.

0740-3232/87/112092-04$02.00

2. GENERAL PROPERTIES OF THE LAYER MATRICES

The matrices in Eqs. (5) are unimodular (have unit determinants). Thus any approximate matrices derived from them will be unimodular, to the appropriate order of approximation. For example, the matrices corresponding to linear $\epsilon(z)$, and taken to second order in δz_n, will be unimodular to second order in δz_n.

The matrix M_n takes (E_n, D_n) into (E_{n+1}, D_{n+1}) according to Eq. (1). For the moment, we write this matrix as $M(n, n+1)$. The inverse matrix takes (E_{n+1}, D_{n+1}) into (E_n, D_n). This is $M(n+1, n)$ so

$$M^{-1}(n, n+1) = M(n+1, n). \tag{7}$$

In other words, taking the inverse of M is equivalent to the exchange of n and $n+1$. Now the inverse of a unimodular two-by-two matrix is

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^{-1} = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}. \tag{8}$$

Thus we have two symmetries. First, we can obtain m_{22} from m_{11} by exchanging n and $n+1$. Second, each of the off-diagonal elements is antisymmetric with respect to this interchange.

At *normal incidence* there is no physical distinction between s and p polarizations, and $r_p = r_s$. From Eqs. (3) and (4), on using the normal-incidence limiting values, $q_a \to n_a\omega/c$, $q_b \to n_b\omega/c$, $Q_a \to n_a^{-1}\omega/c$, and $Q_b \to n_b^{-1}\omega/c$, where $n = \epsilon^{1/2}$ is the refractive index, we find the relationships

$$p_{11} = s_{22}, \qquad p_{12} = -c^2/\omega^2 s_{21},$$
$$p_{21} = \omega^2/c^2 s_{12}, \qquad p_{22} = s_{11}. \tag{9}$$

3. CALCULATION OF THE MATRIX ELEMENTS

In this section we derive general expressions for the s and p matrix elements for an arbitrary profile. These expressions are then evaluated, to third order in δz_n, for the linear and cubic profiles.

Expressions up to the second order in δz_n were derived in TR, Sec. 12-5, by using matrix methods. An equivalent method that is due to Brekhovskikh[4] (explained in Sec. 5-5 of TR) is somewhat simpler to apply and will be used here. The reflection amplitude r_s for an inhomogeneity extending from a to b is calculated in the form

$$r_s = \exp(2iq_a a)\,\frac{q_a u(a) - q_b v(a)}{q_a u(a) + q_b v(a)}, \tag{10}$$

where $u(z)$ and $v(z)$ satisfy the coupled integral equations

$$u(z) = 1 - iq_b \int_z^b d\zeta v(\zeta), \qquad v(z) = 1 - iq_b^{-1} \int_z^b d\zeta q^2(\zeta) u(\zeta). \tag{11}$$

These equations may be iterated to give $u = \sum u_n$ and $v = \sum v_n$, starting with $u_0 = 1 = v_0$. The nth-order iterates are nth order in the interfacial thickness. They are given, for $n \geq 1$, by

$$u_n(z) = -iq_b \int_z^b d\zeta v_{n-1}(\zeta),$$
$$v_n(z) = -iq_b^{-1} \int_z^b d\zeta q^2(\zeta) u_{n-1}(\zeta). \tag{12}$$

To evaluate r_s to third order we need $u_n(a)$ and $v_n(a)$ up to $n = 3$. The resulting $u(a)$ and $v(a)$ are

$$u(a) = 1 - iq_b(b-a) - \int_a^b dz q^2(z)(z-a)$$
$$+ iq_b \int_a^b dz q^2(z)(z-a)(b-z) + \ldots, \tag{13}$$

$$v(a) = 1 - i/q_b \int_a^b dz q^2(z) - \int_a^b dz q^2(z)(b-z)$$
$$+ i/q_b \int_a^b dz q^2(z) \int_a^z d\zeta q^2(\zeta)(z-\zeta) + \ldots. \tag{14}$$

When $\epsilon(z)$ is real (no absorption), so is $q^2 = \epsilon\omega^2/c^2 - K^2$, and thus for even n all u_n and v_n are real, and for odd n they are imaginary. We write $u(a) = u_r + iu_i$ and $v(a) = v_r + iv_i$, substitute into Eq. (10), and compare with Eq. (3). This leads to the identifications

$$s_{11} = v_r, \qquad s_{12} = -u_i/q_b,$$
$$s_{21} = q_b v_i, \qquad s_{22} = u_r. \tag{15}$$

The p-polarization matrix elements may be obtained similarly. The reflection amplitude is calculated in the form

$$-r_p = \exp(2iq_a a)\,\frac{Q_a U(a) - Q_b V(a)}{Q_a U(a) + Q_b V(a)}, \tag{16}$$

where now $U(z)$ and $V(z)$ satisfy the coupled integral equations

$$U(z) = 1 - iQ_b \int_z^b d\zeta \epsilon(\zeta) V(\zeta),$$
$$V(z) = 1 - iQ_b^{-1} \int_z^b d\zeta \epsilon^{-1}(\zeta) q^2(\zeta) U(\zeta). \tag{17}$$

The results of iterating up to third order in the thickness are

$$U(a) = 1 - iQ_b \int_a^b dz \epsilon(z) - \int_a^b dz \epsilon(z)$$
$$\times \int_z^b d\zeta q^2(\zeta)/\epsilon(\zeta) + iQ_b \int_a^b dz \epsilon(z)$$
$$\times \int_z^b d\zeta q^2(\zeta)/\epsilon(\zeta) \int_\zeta^b d\xi \epsilon(\xi) + \ldots, \tag{18}$$

$$V(a) = 1 - iQ_b^{-1} \int_a^b dz q^2(z)/\epsilon(z)$$
$$- \int_a^b dz q^2(z)/\epsilon(z) \int_z^b d\zeta \epsilon(\zeta) + \ldots$$
$$+ iQ_b^{-1} \int_a^b dz q^2(z)/\epsilon(z) \int_z^b d\zeta \epsilon(\zeta)$$
$$\times \int_\zeta^b d\zeta q^2(\xi)/\epsilon(\xi) + \ldots. \tag{19}$$

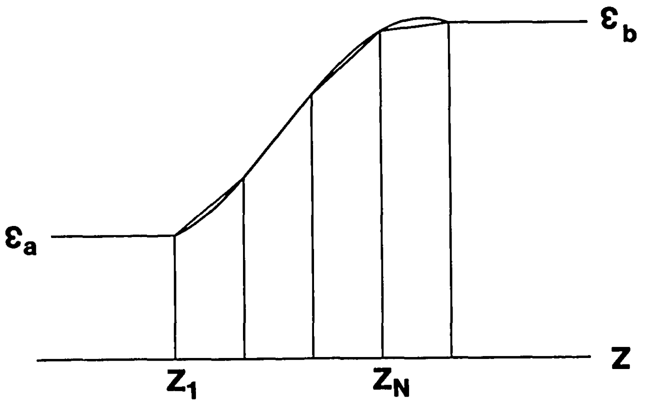

Fig. 1. A stack of N nonuniform layers bounded by media with dielectric constants ϵ_a and ϵ_b. The nth layer extends from z_n to z_{n+1}, and in this case the dielectric function within the nth layer is approximated by the line passing through $\epsilon(z_n)$ and $\epsilon(z_{n+1})$ (linear fit). The cubic fit, discussed in Section 5, is indistinguishable from the exact $\epsilon(z)$ on this scale.

We again write $U(a) = U_r + iU_i$, $V(a) = V_r + iV_i$, substitute into Eq. (16), and compare with Eq. (4). This leads to

$$p_{11} = V_r, \qquad p_{12} = -U_i/Q_b,$$
$$p_{21} = Q_b V_i, \qquad p_{22} = U_r. \tag{20}$$

4. THIRD-ORDER RESULTS FOR A LINEAR FIT TO $\epsilon(z)$

The above expressions are for a general stratification extending from $z = a$ to $z = b$. We now specialize to a profile that has $\epsilon(z)$ linear in z [as given by Eq. (6)] and extends from z_n to z_{n+1}. An arbitrary profile can then be approximated by a set of such layers, as illustrated in Fig. 1.

The matrix elements for the s polarization are, to third order,

$$s_{11} = 1 + (\delta z_n)^2[K^2/2 - \omega^2/c^2(2\epsilon_n + \epsilon_{n+1})/6],$$
$$s_{12} = \delta z_n + (\delta z_n)^3[K^2/6 - \omega^2/c^2(\epsilon_n + \epsilon_{n+1})/12],$$
$$s_{21} = \delta z_n[K^2 - \omega^2/c^2(\epsilon_n + \epsilon_{n+1})/2]$$
$$+ (\delta z_n)^3[K^4/6 - K^2\omega^2/c^2(\epsilon_n + \epsilon_{n+1})/6$$
$$+ \omega^4/c^4(\epsilon_n^2 + 3\epsilon_n\epsilon_{n+1} + \epsilon_{n+1}^2/30],$$
$$s_{22} = 1 + (\delta z_n)^2[K^2/2 - \omega^2/c^2(\epsilon_n + 2\epsilon_{n+1})/6]. \tag{21}$$

The corresponding results for the p polarization are

$$p_{11} = 1 + (\delta z_n)^2\{K^2[2(\epsilon_{n+1}^2/\delta\epsilon_n)\log(\epsilon_{n+1}/\epsilon_n)$$
$$- \epsilon_n - \epsilon_{n+1}]/4\delta\epsilon_n - \omega^2/c^2(\epsilon_n + 2\epsilon_{n+1})/6\},$$
$$p_{12} = \delta z_n(\epsilon_n + \epsilon_{n+1})/2 + (\delta z_n)^3\{K^2[\epsilon_{n+1}^4 - \epsilon_n^4 - 4\epsilon_n^2\epsilon_{n+1}^2$$
$$\times \log(\epsilon_{n+1}/\epsilon_n)]/16(\delta\epsilon_n)^3$$
$$- \omega^2/c^2(\epsilon_n^2 + 3\epsilon_n\epsilon_{n+1} + \epsilon_{n+1}^2)/30\},$$
$$p_{21} = \delta z_n[K^2 \log(\epsilon_{n+1}/\epsilon_n)/\delta\epsilon_n - \omega^2/c^2]$$
$$+ (\delta z_n)^3\{\omega^4/c^4(\epsilon_n + \epsilon_{n+1})/12 + K^2\omega^2/c^2$$
$$\times [\delta\epsilon_n(\epsilon_n^2 + 10\epsilon_n\epsilon_{n+1} + \epsilon_{n+1}^2) - 6(\epsilon_n^3 + \epsilon_{n+1}^3)$$
$$\times \log(\epsilon_{n+1}/\epsilon_n)]/36(\delta\epsilon_n)^3 + K^4[(\epsilon_{n+1}^2 + \epsilon_n^2)\log(\epsilon_{n+1}/\epsilon_n)$$
$$- (\epsilon_{n+1}^2 - \epsilon_n^2)]/4(\delta\epsilon_n)^3\},$$
$$p_{22} = 1 + (\delta z_n)^2\{K^2[\epsilon_n + \epsilon_{n+1} - (2\epsilon_n^2/\delta\epsilon_n)\log(\epsilon_{n+1}/\epsilon_n)]/4\delta\epsilon_n$$
$$- \omega^2/c^2(2\epsilon_n + \epsilon_{n+1})/6\}. \tag{22}$$

For numerical work it is faster and more accurate to replace the terms containing $\log(\epsilon_{n+1}/\epsilon_n)$ by the leading terms in the expansion in terms of $\delta\epsilon_n/\epsilon_n$. We obtain

$$p_{11} \simeq 1 + (\delta z_n)^2[K^2(2\epsilon_n + \epsilon_{n+1})/6\epsilon_n - \omega^2/c^2(\epsilon_n + 2\epsilon_{n+1})/6],$$
$$p_{12} \simeq \delta z_n(\epsilon_n + \epsilon_{n+1})/2 + (\delta z_n)^3[K^2(\epsilon_n + \epsilon_{n+1})/12$$
$$- \omega^2/c^2(\epsilon_n^2 + 3\epsilon_n\epsilon_{n+1} + \epsilon_{n+1}^2)/30],$$
$$p_{21} \simeq \delta z_n[K^2(\epsilon_n^{-1} + \epsilon_{n+1}^{-1})/2 - \omega^2/c^2]$$
$$+ (\delta z_n)^3[\omega^4/c^4(\epsilon_n + \epsilon_{n+1})/12 - K^2(\omega^2/c^2)/3$$
$$+ K^4(\epsilon_n^{-1} + \epsilon_{n+1}^{-1})/12],$$
$$p_{22} \simeq 1 + (\delta z_n)^2[K^2(\epsilon_n + 2\epsilon_{n+1})/6\epsilon_{n+1} - \omega^2/c^2(2\epsilon_n + \epsilon_{n+1})/6]. \tag{23}$$

5. THIRD-ORDER RESULTS FOR A CUBIC FIT TO $\epsilon(z)$

The linear fit to $\epsilon(z)$ in $[z_n, z_{n+1}]$ uses just ϵ_n and ϵ_{n+1}. A more accurate fit can be obtained if the derivatives ϵ_n' and ϵ_{n+1}' are known. The four values ϵ_n, ϵ_{n+1}, ϵ_n', and ϵ_{n+1}' are sufficient for a cubic approximation to $\epsilon(z)$ [see TR, Eq. (13.13)]:

$$\epsilon(z) \simeq \epsilon_n + (z - z_n)\epsilon_n' + \left(\frac{z - z_n}{\delta z_n}\right)^2[3\delta\epsilon_n - \delta z_n(2\epsilon_n' + \epsilon_{n+1}')]$$
$$+ \left(\frac{z - z_n}{\delta z_n}\right)^3[\delta z_n(\epsilon_n' + \epsilon_{n+1}') - 2\delta\epsilon_n]. \tag{24}$$

The resulting s wave matrix elements may be found from Eqs. (13) and (14):

$$s_{11} = 1 + (\delta z_n)^2\{K^2/2 - \omega^2/c^2[(7\epsilon_n + 3\epsilon_{n+1})/20$$
$$+ \delta z_n(\epsilon_n'/20 - \epsilon_{n+1}'/30)]\},$$
$$s_{12} = \delta z_n + (\delta z_n)^3\{K^2/6 - \omega^2/c^2[(\epsilon_n + \epsilon_{n+1})/12$$
$$+ \delta z_n(\epsilon_n' - \epsilon_{n+1}')/60]\},$$
$$s_{21} = \delta z_n\{K^2 - \omega^2/c^2[(\epsilon_n + \epsilon_{n+1})/2$$
$$+ \delta z_n(\epsilon_n' - \epsilon_{n+1}')/12]\} + (\delta z_n)^3\{K^4/6$$
$$- K^2\omega^2/c^2[(\epsilon_n + \epsilon_{n+1})/6 + \delta z_n(\epsilon_n' - \epsilon_{n+1}')/40]$$
$$+ \omega^4/c^4[74(\epsilon_n + \epsilon_{n+1})^2 + 124\epsilon_n\epsilon_{n+1}$$
$$+ \delta z_n(26\epsilon_n'\epsilon_n + 37\epsilon_n'\epsilon_{n+1} - 37\epsilon_n\epsilon_{n+1}'$$
$$- 26\epsilon_{n+1}'\epsilon_{n+1}) + (\delta z_n)^2(2\epsilon_n'^2 - 5\epsilon_n'\epsilon_{n+1}'$$
$$+ 2\epsilon_{n+1}'^2)]/2520\},$$
$$s_{22} = 1 + (\delta z_n)^2\{K^2/2 - \omega^2/c^2[(3\epsilon_n + 7\epsilon_{n+1})/20$$
$$+ \delta z_n(\epsilon_n'/30 - \epsilon_{n+1}'/20)]\}. \tag{25}$$

The elements p_{ij} of the p-polarization matrices will not be given. They are complicated, and (as we explain in Section 6) the cubic fit to $\epsilon(z)$ gave results that were close to the simpler linear fit, except in one circumstance.

6. COMPARISON OF THE NUMERICAL TECHNIQUES

In Chap. 13 of TR comparison is made between L1 and L2, the linear approximations taken to first and second order in δz_n. Here we compare L1, L2, and L3. The cubic approximations, C1–C3, were found in general to be similar to the corresponding linear approximations L1–L3, except for profiles thin in comparison with the wavelength, i.e., for $(\omega/c)\Delta z$ small. In that case only a small number of matrices is required, and the cubic approximation is more accurate. In general, however, we have found that a cubic fit to $\epsilon(z)$, which involves the calculation of the derivative of $\epsilon(z)$ at each boundary, is more trouble than it is worth.

The remainder of this section is restricted to comparison of the linear approximations. In all cases we give the fractional errors, with sign, expressed as parts per thousand, the quantity displayed being $10^3(A/E - 1)$, where A and E are the approximate and the exact values, respectively. The dielectric constants $\epsilon_a = 1$ and $\epsilon_b = (4/3)^2$ are used in all three tables. Table 1 compares L1, L2, and L3 for the Rayleigh profile, defined by $\epsilon = \epsilon_a$ for $z < a$, $\epsilon = \epsilon_b$ for $z > b$, and, for $a \leq z \leq b$,

$$\epsilon(z) = \left[(\epsilon_a^{-1/2} + \epsilon_b^{-1/2})/2 + (z - \bar{z})\frac{\epsilon_b^{-1/2} - \epsilon_a^{-1/2}}{\Delta z}\right]^{-2}, \quad (26)$$

where $\bar{z} = (z_a + z_b)/2$ and $\Delta z = b - a$. Figures 2-15 and 5-4 of TR show the normal-incidence reflectivity for the exactly solvable Rayleigh profile.

The next two tables compare L2 and L3 for the exponential profile, given by $\epsilon = \epsilon_a$ for $z < a$, $\epsilon = \epsilon_b$ for $z > z_b$, and, for $z_a \leq z \leq z_b$,

$$\epsilon(z) = (\epsilon_a \epsilon_b)^{1/2} \exp\left[\frac{z - \bar{z}}{\Delta z} \log \frac{\epsilon_b}{\epsilon_a}\right], \quad (27)$$

where $\bar{z} = (a + b)/2$ and $\Delta z = b - a$. The s and p reflectivities and the ellipsometric ratio for the exponential profile are displayed in Figs. 6-4—6-6 of TR. Table 2 shows errors in

Table 1. Fractional Errors (parts per thousand) in the Normal-Incidence Reflectivity for the Rayleigh Profile as a Function of the Interface Thickness Δz^a

$\frac{\omega \Delta z}{c}$	L1	L2	L3
0.2	−0.3	0.004	0.007
0.4	−1	−0.02	0.03
0.6	−3	−0.2	0.07
0.8	−4	−0.7	0.1
1.0	−4	−2	0.2
1.2	−3	−4	0.3
1.4	1	−9	0.4
1.6	10	−20	0.6
1.8	30	−30	0.7
2.0	70	−50	0.9

[a] The profile has been approximated by 10 layers, in each of which $\epsilon(z)$ varies linearly with z.

Table 2. Parts-per-Thousand Errors in the s and p Reflectivities for the Exponential Profile as a Function of the Angle of Incidence[a]

	R_s		R_p	
θ_a (deg)	L2	L3	L2	L3
0	40	0.4	40	0.4
15	40	0.1	40	0.2
30	4	0.0	−30	−0.7
45	−10	−0.02	−10	−1
60	−4	−0.004	−1	−0.7
75	−1	−0.0006	−0.5	−0.2
90	0	0	0	0

[a] The layer thickness is approximately one half of the wavelength in medium a, $(\omega/c)\Delta z = 3$, and $N = 30$.

Table 3. Parts-per-Thousand Errors in the Real and Imaginary Parts of r_p/r_s at 45°, for the Exponential Profile at $(\omega/c)\Delta z = 3$, as a Function of the Number of Layers, N

N	L2	L3
10	600, −20	6, −5
20	650, −3	−0.2, −1
30	70, −1	−0.2, −0.6
40	40, −1	−0.2, −0.3
50	20, −0.5	−0.1, −0.2

R_s and R_p as a function of angle of incidence for fixed thickness and number of layers. Table 3 compares errors in the real and imaginary parts of r_p/r_s, at fixed thickness and angle of incidence, as a function of N, the number of layers.

From the results shown here, and others like them, we conclude that the linear approximation taken to third order in the layer matrix thickness (L3) is generally much better than the previous best matrix approximation L2, which in turn was an improvement over L1. The increased accuracy is obtained at the expense of slightly more complex expressions to program [compare, for example, the s_{12} and s_{21} matrix elements in Eqs. (21), with and without the $(\delta z_n)^3$ terms]. There is a corresponding increase in computer running time: we found that for the same number of layers the L3 method took 30 to 40% longer to run than the L2. This cost in programming and computer time is offset many times over by the up-to-2-orders of magnitude increase in accuracy of the third-order results.

REFERENCES

1. J. Lekner, *Theory of Reflection of Electromagnetic and Particle Waves* (Nijhoff, Dordrecht, The Netherlands, 1987).
2. R. Jacobsson, "Inhomogeneous and coevaporated homogeneous films for optical applications," in *Physics of Thin Films*, G. Hass, M. H. Francombe, and R. W. Hoffman, eds. (Academic, New York, 1975), Vol. 8.
3. B. M. Law and D. Beaglehole, "Model calculations of the ellipsometric properties of inhomogeneous dielectric surfaces," J. Phys. D 14, 115–126 (1981).
4. L. M. Brekhovskikh, *Waves in Layered Media*, 2nd ed. (Academic, New York, 1980).