

Comparison of hyperbolic and hyperboloid conductor electrostatics

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Abstract

The potentials and fields of hyperbolic and hyperboloidal conductors are available analytically. A detailed comparison of the two-dimensional and three-dimensional problems shows strong similarities, but also interesting differences. The electric field near a hyperboloidal needle is stronger (*ceteris paribus*) than near a hyperbolic blade, and dies off faster. The field at the hyperbolic conductor varies as the $1/3$ power of the local curvature. At the hyperboloid conductor the field varies as the $1/4$ power of the local Gaussian curvature (which is the product of the two principal curvatures).

(Some figures in this article are in colour only in the electronic version)

1. Introduction

An experimenter's question, 'what would be the electric field between two razor blades, as a function of the radius of curvature of the blade edges and of their separation?', led to a paper on the electrostatics of hyperbolic conductors [1]. The referee kindly suggested a generalization to three dimensions, using confocal hyperbolic coordinates. It happens that the electrostatics of hyperboloidal conductors (whose boundary is the surface of revolution of a hyperbola) is a solved problem, although one finds it under titles such as 'Fields currents from points' [2], 'Prolate spheroidal harmonics' [3] or 'Taylor cone and jetting from liquid droplets in electrospinning of nanofibers' [4].

The purpose of this paper is to compare the electrostatics of hyperbolic and of hyperboloidal conductors. We present the methods and results in the two-dimensional and three-dimensional cases together, in the hope that this juxtaposition will make an instructive comparison in the teaching of electrostatics.

We shall restrict the discussion to that of a hyperbolic or hyperboloid of revolution electrode above a plane conducting surface, for simplicity, as shown in figure 1. However, all the hyperbolic or hyperboloidal surfaces are equipotentials, and there is an infinity of configuration of electrodes that are solved by these methods. This is illustrated in figure 2 of [1].

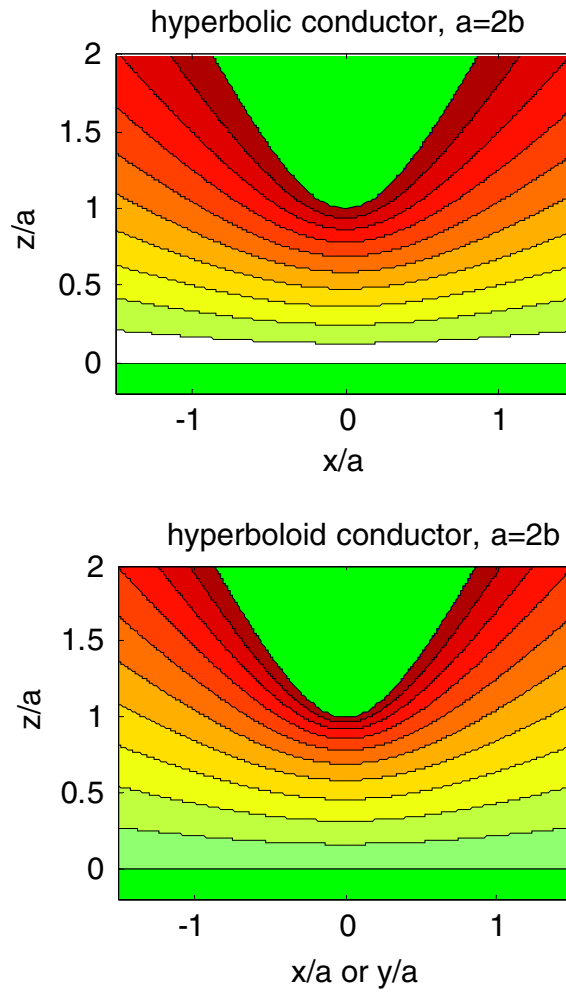


Figure 1. Equipotentials for hyperbolic (upper diagram) and hyperboloid (lower diagram) conductors, at potential V_0 , above an earthed conducting plane. The equipotentials are also hyperbolae and hyperboloids of revolution, respectively, and are drawn at potential steps of $V_0/10$ (i.e. at 90%, 80%, ..., 10% of V_0). Note that the three-dimensional equipotentials are more closely spaced near the tip, and are further apart near the earthed plane, than the two-dimensional equipotentials.

2. Summary of the two-dimensional results

We adapt the results of [1] for comparison with the usual three-dimensional geometry, in which the electrode axis is the z -axis. That is, we take the hyperbolic conductor to be

$$\frac{z^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{or} \quad \{z = a \cosh u, x = b \sinh u\}. \quad (1)$$

Then the equipotentials are hyperbolae, parametrized by

$$\{z = a' \cosh u, x = b' \sinh u\}, \quad a' = a \cos v - b \sin v, \quad b' = a \sin v + b \cos v. \quad (2)$$

Note that (a', b') are obtained from (a, b) by a 'rotation' through angle ν , so

$$a'^2 + b'^2 = a^2 + b^2 \equiv c^2. \quad (3)$$

Thus the equipotential hyperbolae are confocal, with foci at $\pm c$.

Let β represent the half-angle of the cone formed by the asymptotes $z = \frac{a}{b}|x|$ of the hyperbolic conductor, and α be its complement:

$$\beta = \arctan\left(\frac{b}{a}\right), \quad \alpha = \arctan\left(\frac{a}{b}\right), \quad \alpha + \beta = \frac{\pi}{2}. \quad (4)$$

Then, with $r^2 = x^2 + z^2$, the potential function v is given by

$$v(x, z) = \frac{1}{2} \arccos \left\{ \frac{r^2}{c^2} - \sqrt{\left(\frac{r^2}{c^2} + 1\right)^2 - \frac{4z^2}{c^2}} \right\} - \beta. \quad (5)$$

This is zero on the hyperbola (1) and α on the plane $z = 0$, so if the conductor is at potential V_0 and the plane is earthed, the physical potential is $V(x, z) = [\alpha - v(x, z)]V_0/\alpha$.

The field lines are arcs of ellipses, with semiaxes $c \cosh u$, $c \sinh u$:

$$\left(\frac{z}{c \cosh u}\right)^2 + \left(\frac{x}{c \sinh u}\right)^2 = 1. \quad (6)$$

The components of the electric field are $E_x = -\partial_x V$, $E_z = -\partial_z V$. The electric field intensity is

$$E^2(x, z) = \frac{(V_0/\alpha)^2}{c^2 C^2 - z^2/C^2} \quad (7)$$

$$C^2 = \frac{1}{2} + \frac{1}{2} \left\{ \frac{r^2}{c^2} + \left[\left(\frac{r^2}{c^2} + 1\right)^2 - 4\frac{z^2}{c^2} \right]^{\frac{1}{2}} \right\}.$$

In the $x = 0$ symmetry plane $C = 1$ and the maximum field strength is at the apex of the hyperbola ($z = a$); the minimum is at the earthed plane ($z = 0$):

$$E(0, z) = \frac{V_0/\alpha}{[c^2 - z^2]^{\frac{1}{2}}}. \quad (8)$$

The field lines are normal to the equipotential surfaces. At $z = 0$ the magnitude of the electric field is

$$E(x, 0) = \frac{V_0/\alpha}{[c^2 + x^2]^{\frac{1}{2}}}. \quad (9)$$

On the hyperbolic conductor specified by (1) the field strength is

$$E = \frac{V_0/\alpha}{[b^2 + c^2 x^2/b^2]^{\frac{1}{2}}} = \frac{V_0/\alpha}{[b^2 + c^2 \sinh^2 u]^{\frac{1}{2}}}. \quad (10)$$

A relationship between the electric field strength at a point on a conductor surface and the local value of the curvature of the surface cannot exist in general [5, 1], since the field depends on the placement and electric potential of other conductors, whereas the curvature is a purely local geometric property. Nevertheless, there can be a relationship between E and the curvature κ in situations of high symmetry, such as an isolated sphere (for which $E \sim \kappa$) or families of confocal hyperbolae. In [1], it is shown that the field on the surface of the hyperbola (1) at potential V_0 is proportional to the cube root of the local value of the curvature:

$$E = \left(\frac{|\kappa|}{ab}\right)^{\frac{1}{3}} V_0. \quad (11)$$

In general there are two curvatures at a point on a surface in three dimensions, the principal curvatures κ_1 and κ_2 [6]. For the sphere these are equal, and for a hyperbolic ‘knife edge’ conductor one curvature is zero and the other is (see section 6 of [1])

$$\kappa = \frac{ab^4}{[b^4 + c^2x^2]^{\frac{3}{2}}}. \quad (12)$$

In the three-dimensional case relationships like (11) between the electric field strength and the principal curvatures of the hyperboloid will also be shown to hold.

3. Electrostatics of hyperboloid conductors

We now consider hyperboloidal electrodes, which are surfaces of revolution, formed by rotating hyperbolae such as those defined in (1) and (2) about the z -axis. The rotation of the hyperbola (1) gives the hyperboloid

$$\frac{z^2}{a^2} - \frac{\rho^2}{b^2} = 1 \quad \text{or} \quad \{z = a \cosh u, \rho = b \sinh u\}, \quad (13)$$

where $\rho = (x^2 + y^2)^{\frac{1}{2}}$ is the distance from the axis of symmetry. The three-dimensional problem is *not* solved by the substitution of ρ for x in the results of the previous section: the Laplacian in (ρ, z, ϕ) cylindrical coordinates is

$$\nabla^2 = \partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2 + \rho^{-2}\partial_\phi^2. \quad (14)$$

In terms of the prolate spheroidal (or confocal hyperbolic or hyperbolic–elliptic) coordinates u, v defined by [2–4]

$$z = cuv, \quad \rho = c[(u^2 - 1)(1 - v^2)]^{\frac{1}{2}} \quad (15)$$

(where $c^2 = a^2 + b^2$ as before), the Laplacian in the absence of ϕ -dependence is

$$\nabla^2 = [u^2\partial_u^2 - 2u\partial_u] - [v^2\partial_v^2 - 2v\partial_v]. \quad (16)$$

Separation of variables, with separation of variables constant $\ell(\ell + 1)$, gives the Legendre differential equation for both the functions of u and of v . Thus Laplace’s equation for the potential, $\nabla^2 V(\rho, z) = 0$, is satisfied by

$$V(\rho, z) = \sum_{\ell=0}^{\infty} [A_\ell P_\ell(u) + B_\ell Q_\ell(u)][C_\ell P_\ell(v) + D_\ell Q_\ell(v)]. \quad (17)$$

The inverses of (15) are (now the square of the distance from the origin becomes $r^2 = \rho^2 + z^2$)

$$\begin{aligned} 2u^2 &= \frac{r^2}{c^2} + 1 + \sqrt{\frac{r^4}{c^4} + \frac{2}{c^2}(\rho^2 - z^2) + 1} \\ 2v^2 &= \frac{r^2}{c^2} + 1 - \sqrt{\frac{r^4}{c^4} + \frac{2}{c^2}(\rho^2 - z^2) + 1}. \end{aligned} \quad (18)$$

On the symmetry axis $\rho = 0$, we have $u = 1$ and $v = z/c$. (We take $z^2 < c^2 = a^2 + b^2$ as the region of interest for $\rho = 0$.) Thus the ranges of u and v are $1 \leq u < \infty$ and $-1 \leq v < 1$, respectively. Since $Q_\ell(1)$ diverges to infinity, physical solutions will have $B_\ell = 0$. Further, the $\ell = 0$ solution in (17) suffices for the problem at hand. Since [3] $P_0(v) = 1$, $Q_0(v) = \frac{1}{2} \ln\left(\frac{1+v}{1-v}\right)$, the required potential is $C_0 + D_0 \frac{1}{2} \ln\left(\frac{1+v}{1-v}\right)$. We wish the potential to be zero in the plane $z = 0$, where $u = \sqrt{\rho^2/c^2 + 1}$ and $v = 0$. Thus we take

$C_0 = 0$. On the hyperboloid (13) we have $v = a/c$. Thus the required potential, with values V_0 on the hyperboloid (13) and zero on the plane $z = 0$, is

$$V(\rho, z) = V_0 \frac{\ln\left(\frac{1+v}{1-v}\right)}{\ln\left(\frac{c+a}{c-a}\right)} \equiv U_0 \ln\left(\frac{1+v}{1-v}\right). \quad (19)$$

From (15), we see that fixed values of v (equipotentials) are hyperboloids of revolution: on $v = v_1$ we have

$$\frac{z^2}{c^2 v_1^2} - \frac{\rho^2}{c^2(1-v_1^2)} = 1. \quad (20)$$

Fixed values of u give ellipsoids of revolution (prolate spheroids): on $u = u_1$

$$\frac{z^2}{c^2 u_1^2} + \frac{\rho^2}{c^2(u_1^2 - 1)} = 1. \quad (21)$$

The surfaces (20) and (21) are orthogonal families (the product of their slopes $dz/d\rho$ at intersection is -1), so since (20) represents equipotentials, (21) represents field lines. The components of the electric field are

$$E_\rho = -\partial_\rho V = -(\partial_\rho v) \frac{dV}{dv}, \quad E_z = -\partial_z V = -(\partial_z v) \frac{dV}{dv}. \quad (22)$$

The resulting electric field intensity is

$$E^2(\rho, z) = \frac{8U_0^2 c^2}{(c^2 + s^2 - r^2)s^2}, \quad (23)$$

where s is a length defined by

$$s^4 = r^4 + 2c^2(\rho^2 - z^2) + c^4. \quad (24)$$

On the plane $z = 0$ (the earthed conductor in our example), the field magnitude is

$$E(\rho, 0) = \frac{2U_0}{[c^2 + \rho^2]^{\frac{1}{2}}}. \quad (25)$$

On the hyperboloid conductor at potential $V_0 = U_0 \ln\left(\frac{c+a}{c-a}\right)$, the magnitude of the electric field is

$$E(v = a/c) = \frac{2cU_0}{[b^4 + c^2\rho^2]^{\frac{1}{2}}}. \quad (26)$$

Thus the maximum and minimum field magnitudes on the symmetry axis (i.e. at the apex of the hyperboloid of revolution, and directly below it at the zero-potential surface) are

$$E(0, a) = \frac{2cU_0}{b^2}, \quad E(0, 0) = \frac{2U_0}{c}. \quad (27)$$

The principal curvatures of the hyperboloid of revolution (13) are (see the appendix)

$$\kappa_1 = \frac{ab^4}{[b^4 + c^2\rho^2]^{\frac{3}{2}}}, \quad \kappa_2 = \frac{a}{[b^4 + c^2\rho^2]^{\frac{1}{2}}}. \quad (28)$$

One of these (κ_1) has of course the same form as the curvature of the hyperbolic electrode, given in (12). Comparing (26) and (28) gives the proportionalities $E \sim (\kappa_1)^{\frac{1}{2}}$, $E \sim \kappa_2$ for the magnitude of the electric field at the hyperboloid surface. The differential invariants of the surface are [6, 7] the total curvature $\kappa_1 + \kappa_2$ and the Gaussian curvature $\kappa_1\kappa_2$. The field at the surface is not related to the total curvature, but it does have a power relation with the Gaussian curvature, $E \sim (\kappa_1\kappa_2)^{\frac{1}{4}}$. As noted in section 2, such relationships between curvature and field can exist in situations of high symmetry, but not in general.

4. Comparison of hyperbolic and hyperboloid electrostatics

The two-dimensional and three-dimensional conductors, a knife edge with hyperbolic profile and a conical probe with hyperboloidal shape, differ electrostatically mainly in the greater strength of the electric field near the tip compared to near the edge. The maximum values of the field are (we shall write $E^{(2)}$ and $E^{(3)}$ for two- and three-dimensional results)

$$E_{\max}^{(2)} = \frac{V_0}{\alpha b}, \quad E_{\max}^{(3)} = \frac{2cV_0}{b^2 \ln\left(\frac{c+a}{c-a}\right)}. \quad (29)$$

In the case of blunt conductors ($b \gg a$) these both tend to V_0/a , as expected. The ratio $E_{\max}^{(3)}/E_{\max}^{(2)}$ is always greater than unity:

$$\frac{E_{\max}^{(3)}}{E_{\max}^{(2)}} = \frac{2\alpha c}{b \ln\left(\frac{c+a}{c-a}\right)} = \frac{2\alpha}{\cos \alpha \ln\left(\frac{1+\sin \alpha}{1-\sin \alpha}\right)} = \frac{\pi - 2\beta}{\sin \beta \ln\left(\frac{1+\cos \beta}{1-\cos \beta}\right)}. \quad (30)$$

This ratio increases from close to 1 for blunt conductors ($\alpha = \arctan(a/b) \ll 1$) to very large values for sharp conductors ($\alpha \rightarrow \pi/2$, $\beta = \arctan\frac{b}{a} \ll 1$):

$$1 + \frac{\alpha^2}{3} + O(\alpha^4) \xleftarrow{\alpha \ll 1} \frac{E_{\max}^{(3)}}{E_{\max}^{(2)}} \xrightarrow{\beta \ll 1} \frac{\pi/2}{\beta \ln(2/\beta)} + O\left(\frac{1}{\ln(2/\beta)}\right). \quad (31)$$

As a consequence, the three-dimensional field must also decrease more rapidly with distance from the tip than does the two-dimensional field with distance from the edge (*ceteris paribus*), since the path integral of $\mathbf{E} \cdot d\mathbf{l}$ taken between the electrodes at V_0 and the earthed conductor at $z = 0$ must equal V_0 in all cases. The magnitudes of the field in the symmetry plane $x = 0$ and along the symmetry axis $\rho = 0$ are respectively

$$E^{(2)}(0, z) = \frac{V_0}{\alpha[c^2 - z^2]^{\frac{1}{2}}}, \quad E^{(3)}(0, z) = \frac{2cV_0}{(c^2 - z^2) \ln\left(\frac{c+a}{c-a}\right)}. \quad (32)$$

These are equal at $z = z_c = c\left\{1 - \left[2\alpha/\ln\left(\frac{c+a}{c-a}\right)\right]^2\right\}^{\frac{1}{2}}$. For $z < z_c$ the two-dimensional field is larger. At the origin (i.e. on the earthed conductor, directly below the edge or tip) we have

$$\frac{E^{(3)}(0, 0)}{E^{(2)}(0, 0)} = \frac{2\alpha}{\ln\left(\frac{1+\sin \alpha}{1-\sin \alpha}\right)} < 1. \quad (33)$$

The limiting values are

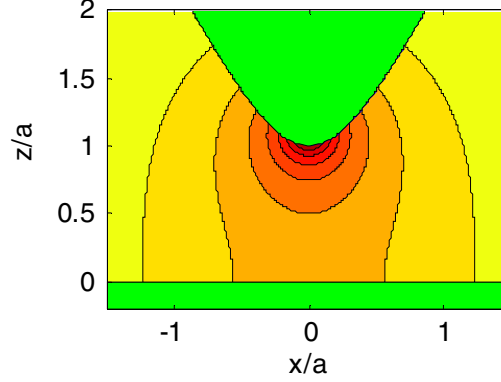
$$1 - \frac{\alpha^2}{6} + O(\alpha^4) \xleftarrow{\alpha \ll 1} \frac{E^{(3)}(0, 0)}{E^{(2)}(0, 0)} \xrightarrow{\beta \ll 1} \frac{\pi/2}{\ln(2/\beta)} + O\left(\frac{\beta}{\ln(2/\beta)}\right). \quad (34)$$

The field strengths in the two- and three-dimensional cases are shown in figure 2.

5. Discussion

Explicit analytical expressions for the potential and field of hyperbolic and hyperboloidal conductors have been presented. The three-dimensional fields are stronger near the tip and weaker near the planar electrode than the corresponding two-dimensional fields. This is in accord with intuition based on density of field lines near the tip and blade, respectively, spreading transversely in both the x and y directions or only in the x direction. Perhaps unexpected is the fact that the fields at the conductors have the same form of functional dependence: compare equations (9) and (25) for the fields at the earthed plane $z = 0$, and (10) and (26) for the fields at the hyperbola and hyperboloid. Thus for the same values of a

electric field magnitude, hyperbolic conductor



electric field magnitude, hyperboloid conductor

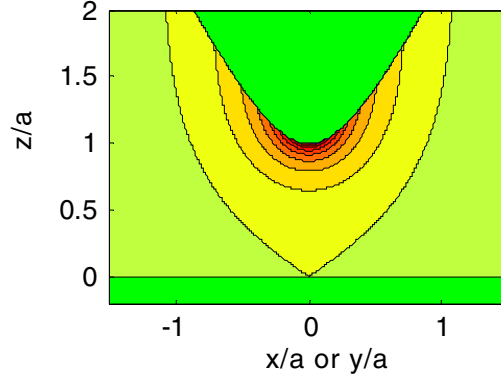


Figure 2. Both upper (hyperbolic conductor) and lower (hyperboloid conductor) diagrams show contours of electric field magnitude; the plots are for $a = 2b$, as in figure 1. The contours are at $0.9E_{\max}$, $0.8E_{\max}$, etc. In the hyperbolic case, the outermost contour is at $0.3E_{\max}$; at the origin the field is $E(0, 0) = (b/c)E_{\max} = E_{\max}/\sqrt{5} \approx 0.447E_{\max}$. For the hyperboloid of revolution the outermost contour is at $0.2E_{\max}$. This is also the value at the origin: $E(0, 0) = (b/c)^2 E_{\max} = E_{\max}/5$.

and b and with $x = \rho$, the two- and three-dimensional fields (and hence the respective charge densities) are in constant ratio, everywhere on the conductors.

The field and charge density on the hyperbolic conductor are proportional to the cube root of the local value of the curvature, $E \sim (\kappa_1)^{\frac{1}{3}}$. For the hyperboloid conductor we find $E \sim (\kappa_1 \kappa_2)^{\frac{1}{4}}$. These relations are interesting in contradicting the naive expectation of simple proportionality of field strength to local curvature. In general however one should not expect any local relationship between surface curvature and field strength, only between the surface value of the logarithmic derivative of the field with respect to distance from the surface and the total curvature [8, 5]:

$$-\frac{1}{E} \frac{\partial E}{\partial n} = \kappa_1 + \kappa_2. \quad (35)$$

Since one curvature is zero in the two-dimensional case and positive in the three-dimensional case, while the other (also positive) curvature is common to the two cases, the fractional rate

of decrease of the field is faster for hyperboloidal than it is for hyperbolic conductors, as we found.

Acknowledgments

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Appendix. Curvatures of a surface

The differential invariants of a surface are the sum and the product of the principal curvatures [6, 7], known respectively as the total and Gaussian curvatures. If the surface is parametrized as $[x, y, z(x, y)]$ (the Monge representation), these are

$$\kappa_1 + \kappa_2 = \frac{[1 + (\partial_x z)^2] \partial_y^2 z + [1 + (\partial_y z)^2] \partial_x^2 z - 2(\partial_x z)(\partial_y z)(\partial_x \partial_y z)}{[1 + (\partial_x z)^2 + (\partial_y z)^2]^{\frac{3}{2}}}, \quad (\text{A.1})$$

$$\kappa_1 \kappa_2 = \frac{(\partial_x^2 z)(\partial_y^2 z) - (\partial_x \partial_y z)^2}{[1 + (\partial_x z)^2 + (\partial_y z)^2]^2}. \quad (\text{A.2})$$

For surfaces of revolution we have $z(x, y) = z(\rho)$, $\rho^2 = x^2 + y^2$. Then the total and Gaussian curvatures reduce to (we use the shorthand $z' = dz/d\rho$, $z'' = d^2z/d\rho^2$)

$$\kappa_1 + \kappa_2 = \frac{\rho^{-1} z' [1 + (z')^2] + z''}{[1 + (z')^2]^{\frac{3}{2}}}, \quad (\text{A.3})$$

$$\kappa_1 \kappa_2 = \frac{\rho^{-1} z' z''}{[1 + (z')^2]^2}. \quad (\text{A.4})$$

For the hyperboloid of revolution $z = \frac{a}{b} [b^2 + \rho^2]^{\frac{1}{2}}$, and the sum and product of the curvatures become

$$\kappa_1 + \kappa_2 = \frac{a(2b^4 + c^2 \rho^2)}{[b^4 + c^2 \rho^2]^{\frac{3}{2}}}, \quad (\text{A.5})$$

$$\kappa_1 \kappa_2 = \frac{a^2 b^4}{[b^4 + c^2 \rho^2]^2}. \quad (\text{A.6})$$

These formulae are in accord with (28).

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