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Reply to 'Comment on "TM, TE and 'TEM' beam modes: exact solutions and their problems" '

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Abstract
Some of the set of exact beam wavefunctions have logarithmically divergent energy integrals, which limits their usefulness to the region close to the beam axis.

Keywords: Laser beams, nonparaxial modes, exact solutions

Solutions of the Helmholtz equation, $(\nabla^2 + k^2)\Psi = 0$, may be used to construct solutions of Maxwell's equations for monochromatic waves (with angular frequency $\omega = ck$) via the vector potential $\boldsymbol{A}$. Sheppard and Saghafi [1–4] have introduced and applied to optical beams the exact solution

$$\Psi_{00} = j_0(kR) = \frac{\sin kR}{kR} \qquad R^2 = \rho^2 + (z - \mathrm{i}b)^2. \tag{1}$$

Ulanowski and Ludlow [5] have noted that this exact solution is just the first of an infinite set built up from spherical Bessel functions and associated Legendre polynomials,

$$\Psi_{\ell m} = j_\ell(kR) P_{\ell m}\left(\frac{z - \mathrm{i}b}{R}\right) \mathrm{e}^{\pm \mathrm{i}m\phi}. \tag{2}$$

Lekner [6] showed that Ψ_{00} gives a divergent normalization integral $\int_0^\infty \mathrm{d}\rho\, \rho |\Psi_{00}|^2$ for the scalar (particle-beam) case, and divergent energy integrals $\int_0^\infty \mathrm{d}\rho\, \rho \overline{u}$ for the TM, TE and 'TEM' beams constructed from Ψ_{00}. [6] also showed that $\Psi_{10} = j_1(kR)(z - \mathrm{i}b)/R$ gives finite energy integrals, as well as finite momentum integrals. In fact, all $\Psi_{\ell m}$ with odd $\ell - m$ give the desired convergence properties.

The question raised by the comment [7] is *do the divergences matter?* For Ψ_{00} the normalization integrand $\rho|\Psi_{00}|^2$ is, in the focal plane $z = 0$, proportional to $\rho \sin^2(k\sqrt{\rho^2 - b^2})/(\rho^2 - b^2) \sim \rho^{-1} \sin^2 k\rho$ for $\rho^2 \gg b^2$. Thus the divergence is logarithmic, as it is for the integral over the (approximate) intensity in high-aperture optical systems, as given in equation (1) of [7]. In the electromagnetic case, again for Ψ_{00}, the energy integrand has asymptotic form proportional to ρ^{-1} (leading to logarithmic divergence), while the momentum integrand has its leading term proportional to $\rho^{-2} \cos k\rho \sin k\rho$, giving the finite integral

$$c \int_0^\infty \mathrm{d}\rho\, \rho \overline{p}_z = \frac{A_0^2}{16\pi} \times \frac{[1 - \frac{1}{2}\beta^{-1}(1 - \mathrm{e}^{-2\beta})][1 - (2\beta + 1)\mathrm{e}^{-2\beta}]}{[1 - \mathrm{e}^{-2\beta}]^2}, \qquad \beta = kb \tag{3}$$

in the TM and TE cases, with a $\beta/\sinh\beta$ factor multiplying Ψ_{00} to normalize it to unity at the origin. (This expression corrects the misprinted equation (40) of [6].) In the Ψ_{10} TM, TE and 'TEM' cases, both the energy and the momentum integrals are finite, as given in [6]; the leading terms in the energy and momentum integrands are $\rho^{-3}(\beta^2 + \sin^2 k\rho)$ and $\rho^{-3} \sin^2 k\rho$, respectively.

One can avoid the divergences associated with the Ψ_{00} wavefunction by using a cut-off or a convergence factor, with characteristic distance from the beam axis ρ_c. But any finite ρ_c omits an infinite amount of electromagnetic energy associated with the exact wavefunction. The usefulness of Ψ_{00} thus appears to be limited to the region near the beam axis, where it is an improvement over the approximate Gaussian wavefunction, especially for values of $kb \lesssim 2$, i.e. in high-aperture situations [8].

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