Matrix methods in reflection and transmission of compressional waves by stratified media

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The techniques used in optics and microwave and radio physics for the easy and efficient calculation of reflection and transmission by stratified media are adapted to acoustic compressional waves. The method involves taking the product of N 2×2 matrices when the stratification is approximated by N layers. These layers can be chosen to have linear variation in the acoustic parameters to best represent the actual stratification without undue complexity in the resulting matrix elements. It is possible to guarantee unimodularity of the matrices, thus making sure that energy conservation and a reciprocity law are automatically satisfied. Accuracy is tested against an exactly solvable model stratification, in which the density and speed of sound both vary exponentially with depth.

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INTRODUCTION

The reflection and transmission of acoustic waves by an arbitrary stratified medium can be calculated by approximating the stratification by a set of uniform layers, and using matrices to connect the sound field across the layers.[1–4] A related methodology is used to obtain impulse responses.[5–9] In the case of media that support shear as well as compressional waves, 4×4 matrices are required; for fluid media the matrices become 2×2. An analogous scheme is used in the calculation of reflection and transmission of electromagnetic waves; for isotropic media there are two sets of 2×2 matrices, one for each of the polarizations of the incident wave. A summary of the various methods used, and references (going back to Rayleigh) may be found in Chaps. 12 and 13 of a recent monograph on reflection.[10] It was noted in Ref. 10 that a minor reformulation of the problem made the matrix elements real in the absence of absorption. This can give a fourfold saving in computation time (other things being equal), since four matrix products must be found to obtain the product of two matrices with complex elements. Further gains in accuracy and efficiency can be made by removing the approximation that each layer is homogeneous, and letting its physical characteristics vary so that there is no discontinuity at the boundaries between the layers.[10,11] An important aspect, not previously noted, is that unimodularity of the matrices guarantees energy conservation and reciprocity. This fact is proved and used in the present paper.

I. ACOUSTIC WAVES IN A PLANAR STRATIFICATION

The linearized equation for the acoustic pressure p (the time-dependent oscillatory variation associated with the acoustic wave) is[12,13]

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{\rho}\nabla\rho\cdot\nabla p = 0, \qquad (1)$$

where c and ρ are the local values of the phase velocity and of the density, respectively. In a planar stratification the velocity and density are functions only of the depth z. For a plane monochromatic wave propagating in the zx plane, solutions of (1) have the form

$$p(z,x,t) = e^{i(Kx-\omega t)}P(z), \qquad (2)$$

where ω is the angular frequency of the wave and K is the x component of the wave vector, which is a constant of the motion. For a planar stratification between uniform media a and b,

$$K = (\omega/c_a)\sin\theta_a = (\omega/c_b)\sin\theta_b, \qquad (3)$$

where θ_a and θ_b are the angles of incidence and refraction. (If grazing angles are used, K is proportional to the cosine of the grazing angle divided by the local speed of sound.)

The differential equation for $P(z)$ is [from (1) and (2)]

$$\rho\frac{d}{dz}\left(\frac{1}{\rho}\frac{dP}{dz}\right) + q^2 P = 0, \qquad (4)$$

where $q(z)$ is the normal component of the wave vector and is given by

$$q^2(z) = \omega^2/c^2(z) - K^2. \qquad (5)$$

In media a and b, q takes the constant values

$$q_a = (\omega/c_a)\cos\theta_a, \quad q_b = (\omega/c_b)\cos\theta_b. \qquad (6)$$

The reflection and transmission amplitudes r and t are defined in terms of the forms of the acoustic pressure in media a and b. Assuming that the sound field is incident from medium a, these are

$$e^{iq_a z} + re^{-iq_a z} \leftarrow P(z) \rightarrow te^{iq_b z}. \qquad (7)$$

The reflectance and transmittance are given in terms of the amplitudes r and t by

$$R = |r|^2, \quad T = (Q_b/Q_a)|t|^2, \qquad (8)$$

where $Q_a = q_a/\rho_a$, $Q_b = q_b/\rho_b$.

The second-order differential equation for $P(z)$ may be written as a pair of coupled first-order differential equations in P and its derivative divided by the density:

$$\frac{1}{\rho}\frac{dP}{dz} = D, \quad \rho\frac{dD}{dz} = -q^2 P. \qquad (9)$$

If one approximates an arbitrary stratification by a set of homogeneous layers (represented by the dashed lines in Fig. 1), ρ and q take the constant values ρ_n and q_n in $z_n \leqslant z \leqslant z_{n+1}$, and the solutions of (9) in the nth layer are

$$P(z) = P_n \cos q_n (z - z_n) + Q_n^{-1} D_n \sin q_n (z - z_n),$$
$$D(z) = D_n \cos q_n (z - z_n) - Q_n P_n \sin q_n (z - z_n), \quad (10)$$

where P_n,D_n are the values of P, D at $z = z_n$, and $Q_n = q_n / \rho_n$.

Now P and D are continuous at discontinuities in c and/or ρ [otherwise (4) would not be satisfied at the discontinuities]; continuity at z_{n+1} gives

$$P_{n+1} = P_n \cos \delta_n + Q_n^{-1} D_n \sin \delta_n,$$
$$D_{n+1} = D_n \cos \delta_n - Q_n P_n \sin \delta_n, \quad (11)$$

where $\delta_n = q_n (z_{n+1} - z_n) \equiv q_n \delta z_n$ is the phase increment across the layer. Thus the vector formed from P_{n+1} and D_{n+1} is related by a matrix to the vector formed from P_n and D_n:

$$\begin{pmatrix} P_{n+1} \\ D_{n+1} \end{pmatrix} = \begin{pmatrix} \cos \delta_n & Q_n^{-1} \sin \delta_n \\ -Q_n \sin \delta_n & \cos \delta_n \end{pmatrix} \begin{pmatrix} P_n \\ D_n \end{pmatrix}. \quad (12)$$

This 2×2 matrix has unit determinant (even in the presence of absorption, when q_n is complex). For real q_n the matrix is real, in contrast to the usual method that makes the off-diagonal elements imaginary. Note also that the matrix is equal to $(-)^{\ell}$ times the unit matrix when the phase increment $\delta_n = \ell\pi$ with integer ℓ. If ℓ is an even integer, the layer has no effect on the sound field (at the particular frequency, thickness, and angle of incidence that make $\delta_n = \ell\pi$); if ℓ is an odd integer, the layer reverses the signs of P and D.

The approximation in which an arbitrary stratification is represented by a set of homogeneous layers leads to matrices such as the one in (12). In the next section we will remove this restriction, and consider representations where ρ and c have arbitrary variation; the example of linear variation is shown in Fig. 2.

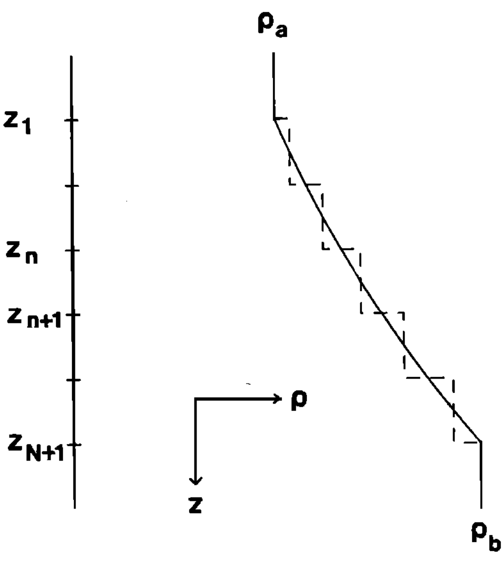

FIG. 1. A stratification with continuously varying density (or speed), approximated by N homogeneous layers. Here $N = 5$.

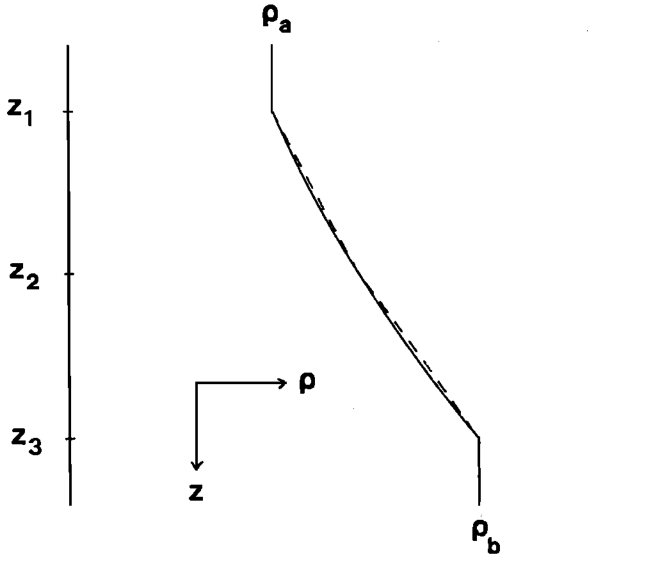

FIG. 2. The same profile as in Fig. 1, approximated by two layers in which the parameters vary linearly with depth z.

II. MATRIX METHODS FOR NONUNIFORM LAYERS

The pair of coupled equations (9) may be written as

$$\frac{dP}{dz} = \rho D, \quad \frac{dD}{dz} = -\frac{q^2}{\rho} P. \quad (13)$$

In $z_n \leqslant z \leqslant z_{n+1}$, the integral versions of (13), incorporating the boundary values at z_n, are

$$P(z) = P_n + \int_{z_n}^{z} d\zeta\, \rho(\zeta) D(\zeta),$$
$$D(z) = D_n - \int_{z_n}^{z} d\zeta \frac{q^2(\zeta) P(\zeta)}{\rho(\zeta)}. \quad (14)$$

The coupled integral equations can be solved by iteration. We set

$$P(z) = \sum_{j=0}^{\infty} P^{(j)}(z), \quad D(z) = \sum_{j=0}^{\infty} D^{(j)}(z), \quad (15)$$

and start with $P^{(0)} = P_n$, $D^{(0)} = D_n$. The superscript j gives the degree of the correction in the thickness $\delta z_n = z_{n+1} - z_n$. The first iterates are

$$P^{(1)}(z) = D_n \int_{z_n}^{z} d\zeta\, \rho(\zeta),$$
$$D^{(1)}(z) = -P_n \int_{z_n}^{z} d\zeta \frac{q^2(\zeta)}{\rho(\zeta)}. \quad (16)$$

The second-order iterates (evaluated at z_{n+1}) are

$$P^{(2)}(z_{n+1}) = \int_{z_n}^{z_{n+1}} dz\, \rho(z) D^{(1)}(z)$$
$$= -P_n \int_{z_n}^{z_{n+1}} dz\, \rho(z) \int_{z_n}^{z} d\zeta \frac{q^2(\zeta)}{\rho(\zeta)},$$
$$D^{(2)}(z_{n+1}) = -\int_{z_n}^{z_{n+1}} dz \frac{q^2(z) P^{(1)}(z)}{\rho(z)}$$
$$= -D_n \int_{z_n}^{z_{n+1}} dz \frac{q^2(z)}{\rho(z)} \int_{z_n}^{z} d\zeta\, \rho(\zeta). \quad (17)$$

To find the matrix relation between P_{n+1} and D_{n+1} and P_n, D_n we evaluate (15) at z_{n+1}. To second order in δz_n, these equations read

$$P_{n+1} = P_n + D_n I_1 - P_n I_2, \quad D_{n+1} = D_n - P_n J_1 - D_n J_2, \tag{18}$$

where

$$I_1 = \int_{z_n}^{z_{n+1}} dz\, \rho(z),$$
$$I_2 = \int_{z_n}^{z_{n+1}} dz\, \rho(z) \int_{z_n}^{z} d\zeta \frac{q^2(\zeta)}{\rho(\zeta)},$$
$$J_1 = \int_{z_n}^{z_{n+1}} dz \frac{q^2(z)}{\rho(z)},$$
$$J_2 = \int_{z_2}^{z_{n+1}} dz \frac{q^2(z)}{\rho(z)} \int_{z_n}^{z} d\zeta\, \rho(\zeta). \tag{19}$$

The second-order matrix relation is thus

$$\begin{pmatrix} P_{n+1} \\ D_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - I_2 & I_1 \\ -J_1 & 1 - J_2 \end{pmatrix} \begin{pmatrix} P_n \\ D_n \end{pmatrix} \equiv M_n \begin{pmatrix} P_n \\ D_n \end{pmatrix}. \tag{20}$$

Note that by interchange of the order of integration J_2 may be written in the form

$$J_2 = \int_{z_n}^{z_{n+1}} dz\, \rho(z) \int_{z}^{z_{n+1}} d\zeta \frac{q^2(\zeta)}{\rho(\zeta)}, \tag{21}$$

so that

$$I_2 + J_2 = I_1 J_1. \tag{22}$$

Thus the determinant of M_n is equal to $1 + I_2 J_2$; this shows that the matrices obtained by iterating P and D to second order in δz_n have a correction to unimodularity of order (δz_n).[4] If we had stopped at the first order, the determinant of M_n would be $1 + I_1 J_1$, so the correction to unimodularity would be of second order in δz_n. The significance of unimodularity will be seen in the next section. Here we note only that symmetrized starting values for the iteration, namely,

$$P^{(0)} = \tfrac{1}{2}(P_n + P_{n+1}), \quad D^{(0)} = \tfrac{1}{2}(D_n + D_{n+1}), \tag{23}$$

improve the unimodularity. To first order in δz_n, (25) gives

$$P_{n+1} = P_n + \tfrac{1}{2}(D_n + D_{n+1}) I_1,$$
$$D_{n+1} = D_n - \tfrac{1}{2}(P_n + P_{n+1}) J_1, \tag{24}$$

so that

$$(1 + I_1 J_1/4) P_{n+1} = (1 - I_1 J_1/4) P_n + D_n I_1,$$
$$(1 + I_1 J_1/4) D_{n+1} = (1 - I_1 J_1/4) D_n - P_n J_1. \tag{25}$$

The cross coupling of P_{n+1} to D_{n+1} in (24) has the effect of introducing "second-order" terms proportional to $I_1 J_1$ in the decoupled relations (25). The corresponding matrix is

$$M_n = \left(1 + \frac{I_1 J_1}{4}\right)^{-1} \begin{pmatrix} 1 - I_1 J_1/4 & I_1 \\ -J_1 & 1 - I_1 J_1/4 \end{pmatrix}, \tag{26}$$

(the prefactor multiplies every element of the matrix). The determinant of this matrix is unity, exactly. Does perfect unimodularity persist to second order in the layer thickness if the symmetric starting values (23) are used? We have, instead of (16),

$$P^{(1)}(z) = \frac{1}{2}(P_n - P_{n+1}) + \frac{1}{2}(D_n + D_{n+1}) \times \int_{z_n}^{z} d\zeta\, \rho(\zeta),$$
$$D^{(1)}(z) = \frac{1}{2}(D_n - D_{n+1}) + \frac{1}{2}(P_n + P_{n+1}) \times \int_{z_n}^{z} d\zeta \frac{q^2(\zeta)}{\rho(\zeta)}. \tag{27}$$

The first equalities in (17) remain valid with the symmetrized starting point, and give

$$P_{n+1} = P_n + D_n I_1 - \tfrac{1}{2}(P_n + P_{n+1}) I_2,$$
$$D_{n+1} = D_n - P_n J_1 - \tfrac{1}{2}(D_n + D_{n+1}) J_2. \tag{28}$$

Note that there is no cross-coupling of P_{n+1} to D_{n+1} in the genuinely second-order equations. The corresponding matrix is

$$M_n = \begin{pmatrix} \dfrac{1 - I_2/2}{1 + I_2/2} & \dfrac{I_1}{1 + I_2/2} \\ \dfrac{-J_1}{1 + J_2/2} & \dfrac{1 - J_2/2}{1 + J_2/2} \end{pmatrix}. \tag{29}$$

That this matrix is exactly unimodular follows from the identity (22).

These matrices will be applied to the numerical evaluation of reflection and transmission amplitudes in Sec. IV. First we will consider how an arbitrary set of layer matrices determines r and t, and some consequent properties.

III. REFLECTION AND TRANSMISSION AMPLITUDES IN TERMS OF THE PROFILE MATRIX

We have seen how to approximate matrices M_n that give P_{n+1}, D_{n+1} in terms of P_n, D_n, for an arbitrary variation of the acoustical parameters in the nth layer, $z_n \leqslant z \leqslant z_{n+1}$. Let z_1 and z_{N+1} be the boundaries of the stratification that is represented by N layers, with uniform media a (for $z < z_1$) and b (for $z > z_{N+1}$) lying on either side. From (7) we see that the values of P and D at z_1 and at z_{N+1} are given by

$$P_1 = e^{i\alpha} + r e^{-i\alpha}, \quad D_1 = iQ_a (e^{i\alpha} - r e^{-i\alpha}), \quad \alpha \equiv q_a z_1,$$
$$P_{N+1} = t e^{i\beta}, \quad D_{N+1} = iQ_b t e^{i\beta}, \quad \beta \equiv q_b z_{N+1}. \tag{30}$$

Now

$$\begin{pmatrix} P_{N+1} \\ D_{N+1} \end{pmatrix} = M_N \begin{pmatrix} P_N \\ D_N \end{pmatrix} = M_N M_{N-1} \begin{pmatrix} P_{N-1} \\ D_{N-1} \end{pmatrix} = \cdots = M \begin{pmatrix} P_1 \\ D_1 \end{pmatrix}, \tag{31}$$

where

$$M \equiv M_N M_{N-1} \cdots M_n \cdots M_2 M_1 \tag{32}$$

is the *profile matrix*, the sequential product of the N layer matrices. Let m_{ij} be the elements of this 2×2 profile matrix. Then from (30) and (31),

$$\begin{pmatrix} t e^{i\beta} \\ iQ_b t e^{i\beta} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} e^{i\alpha} + r e^{-i\alpha} \\ iQ_a e^{i\alpha} - iQ_a r e^{-i\alpha} \end{pmatrix}. \tag{33}$$

Solving for the reflection and transmission amplitudes r and t we find

$$r = e^{2i\alpha} \frac{Q_a Q_b m_{12} + m_{21} + iQ_a m_{22} - iQ_b m_{11}}{Q_a Q_b m_{12} - m_{21} + iQ_a m_{22} + iQ_b m_{11}}, \quad (34)$$

$$t = e^{i(\alpha - \beta)} \frac{2iQ_a \det M}{Q_a Q_b m_{12} - m_{21} + iQ_a m_{22} + iQ_b m_{11}}, \quad (35)$$

where $\det M = m_{11}m_{22} - m_{12}m_{21}$ is the determinant of the profile matrix. (These results closely follow those derived for electromagnetic waves in Ref. 10, Sec. 12-2.) We will show that a conservation law and a reciprocity law are both satisfied when the profile matrix is unimodular, that is $\det M = 1$.

In the absence of dissipation within any part of the system, and also excluding total internal reflection, all q's and Q's are real. No absorption within the stratification also implies that all the matrix elements are real. Then the reflectance $R = |r|^2$ and transmittance $T = (Q_b/Q_a)|t|^2$ are given by

$$R = \frac{(Q_a Q_b m_{12} + m_{21})^2 + (Q_a m_{22} - Q_b m_{11})^2}{(Q_a Q_b m_{12} - m_{21})^2 + (Q_a m_{22} + Q_b m_{11})^2}, \quad (36)$$

$$T = \frac{4Q_a Q_b (\det M)^2}{(Q_a Q_b m_{12} - m_{21})^2 + (Q_a m_{22} + Q_b m_{11})^2}. \quad (37)$$

Since there is no dissipation, the incident intensity must be equal to the sum of the reflected and transmitted intensities, $R + T = 1$. From the formulas for R and T above,

$$R + T = 1 + \frac{4Q_a Q_b \det M(\det M - 1)}{(Q_a Q_b m_{12} - m_{21})^2 + (Q_a m_{22} + Q_b m_{11})^2}. \quad (38)$$

Thus energy conservation requires $\det M = 1$ or $\det M = 0$. In the case of representation by uniform layers, M is a product of unimodular matrices of the type given in (12), so $\det M = 1$. Since $\det M$ is a continuous function of the matrix elements, $\det M = 0$ is excluded in the general case.

Next we compare the reflection and transmission when the wave is incident "from below" (from medium b). Equation (31) still holds, with the same M as before, but now

$$P_1 = t'e^{-i\alpha}, \quad D_1 = -iQ_a t'e^{-i\alpha},$$
$$P_{N+1} = e^{-i\beta} + r'e^{i\beta}, \quad D_{N+1} = -iQ_b e^{-i\beta} + iQ_b r'e^{i\beta}, \quad (39)$$

where r' and t' are the reflection and transmission amplitudes for incidence from medium b. Thus (33) is replaced by

$$\begin{pmatrix} e^{-i\beta} + r'e^{-i\beta} \\ -iQ_b e^{-i\beta} + iQ_b r'e^{i\beta} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} t'e^{-i\alpha} \\ -iQ_a t'e^{-i\alpha} \end{pmatrix}. \quad (40)$$

This leads to

$$r' = e^{-2i\beta} \frac{Q_a Q_b m_{12} + m_{21} - iQ_a m_{22} + iQ_b m_{11}}{Q_a Q_b m_{12} - m_{21} + iQ_a m_{22} + iQ_b m_{11}}, \quad (41)$$

$$t' = e^{i(\alpha - \beta)} \frac{2iQ_b}{Q_a Q_b m_{12} - m_{21} + iQ_a m_{22} + iQ_b m_{11}}. \quad (42)$$

The reciprocity law[14] $Q_a t' = Q_b t$ [which implies the important result that the transmittances $T = (Q_b/Q_a)|t|^2$ and $T' = (Q_a/Q_b)|t'|^2$ are equal, even if there is absorption within the stratification] is seen to be valid on comparing (35) with (42), *provided* $\det M = 1$. Another reciprocity law, valid only in the absence of absorption, is $r' = -t'r^*/t'^*$. This law,[14] which implies that the reflectance is the same from either side, is verified from the equations for r, r', and t' given above, independently of the value of $\det M$.

We have shown that unimodularity of M is necessary for energy conservation and for the reciprocity law $T' = T$. If each layer matrix is unimodular, M will be unimodular, since the determinant of a product of matrices is equal to the product of their determinants. Thus unimodularity of the layer guarantees these laws, and is a desirable characteristic in any approximation scheme. Of course, unimodularity by itself implies nothing about accuracy or efficiency. These will be considered next.

IV. NUMERICAL METHODS BASED ON THE LAYER MATRICES

The elements of the profile matrix determine the reflection and transmission amplitudes, the profile matrix being found as a product of layer matrices. We will consider two classes of approximations: Those based on layers that have constant acoustical properties (labeled C), and those based on layers within which the acoustical properties vary linearly (labeled L). A subscript on C or L will give the order in layer thickness to which the layer matrix has been calculated. The layer matrix in the C_∞ scheme (stratification approximated by homogeneous layers, each layer matrix calculated exactly) has been given in Eq. (12). The corresponding C_1 and C_2 matrices, obtained from (12), are

$$\begin{pmatrix} 1 & \rho_n \delta z_n \\ -q_n^2 \delta z_n/\rho_n & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 - (q_n \delta z_n)^2/2 & \rho_n \delta z_n \\ -q_n^2 \delta z_n/\rho_n & 1 - (q_n \delta z_n)^2/2 \end{pmatrix}. \quad (43)$$

One might ask: Why expand in powers of δz_n, when the exact matrix is known in simple form? The answer is that in approximating a stratification, as in Fig. 1, it is often necessary to use a large number of uniform layers, each thin (with small $\delta_n \equiv q_n \delta z_n$). The elements of the matrices in (43) are then nearly as accurate, and much faster to calculate, than are the exact sinusoidal layer matrix elements in (12), for a layer that in any case only approximates the physical profile.

It is plausible intuitively, on comparison of Figs. 1 and 2, that a more accurate representation of a continuously varying stratification is in terms of layer matrices which allow variation of the acoustical parameters so as to avoid discontinuities at the layer boundaries. The simplest variation is linear, for example,

$$\rho(z) = \rho_n + (z - z_n)\delta\rho_n/\delta z_n, \quad z_n \leqslant z \leqslant z_{n+1}, \tag{44}$$

where $\delta\rho_n = \rho_{n+1} - \rho_n$. The acoustical parameters are the local density $\rho(z)$ and the local speed of sound $c(z)$. The integrals needed for the evaluation of the matrix elements (Sec. III) have integrands $\rho(z)$ and $q^2(z)/\delta(z) = [\omega^2/c^2(z) - K^2]/\rho(z)$. The integrals can be found analytically if, for example, $\rho(z)$ and $c^{-2}(z)$ are taken to vary linearly within a layer. All integrals except I_1 then contain logarithmic functions with arguments such as ρ_{n+1}/ρ_n. For accuracy and speed of computation, these can be expanded in powers of $\delta\rho_n/\rho_n$; the analogous procedure in the case of the electromagnetic p wave is carried out in Ref. 10, p. 244. This long process can be avoided if we assume the function $\Lambda(z) \equiv q^2(z)/\rho(z)$ to be linear (compare Sec. 5.2 of Ref. 15). The corresponding variation in the speed of sound is such that $c^{-2}(z)$ is quadratic in z: Assume (44) and

$$\Lambda(z) = \Lambda_n + (z - z_n)\frac{\delta\Lambda_n}{\delta z_n}, \quad \delta\Lambda_n = \Lambda_{n+1} - \Lambda_n. \tag{45}$$

Then

$$\frac{\omega^2}{c^2} = \frac{\omega^2}{c_n^2} + (\Lambda_n\,\delta\rho_n + \rho_n\,\delta\Lambda_n)\frac{z - z_n}{\delta z_n} + \delta\Lambda_n\,\delta\rho_n\left(\frac{z - z_n}{\delta z_n}\right)^2. \tag{46}$$

When ρ and $\Lambda = q^2/\rho$ are assumed linear within each layer, the evaluation of the integrals I_1, J_1, I_2, J_2 is elementary. We find

$$\begin{aligned}
I_1 &= \delta z_n(\rho_n + \rho_{n+1})/2,\\
J_1 &= \delta z_n(q_n^2/\rho_n + q_{n+1}^2/\rho_{n+1})/2\\
&= \delta z_n(\rho_n Q_n^2 + \rho_{n+1}Q_{n+1}^2)/2,\\
I_2 &= (\delta z_n)^2[3q_n^2 + 5q_n^2\rho_{n+1}/\rho_n\\
&\quad + q_{n+1}^2\rho_n/\rho_{n+1} + 3q_{n+1}^2]/24,\\
J_2 &= (\delta z_n)^2[3q_n^2 + 5\rho_n q_{n+1}^2/\rho_{n+1}\\
&\quad + \rho_{n+1}q_n^2/\rho_n + 3q_{n+1}^2]/24.
\end{aligned} \tag{47}$$

We note that (22) is satisfied, and that the second-order integrals may be written as

$$\begin{aligned}
I_2 &= I_1J_1/2 + (\delta z_n)^2\rho_n\rho_{n+1}(Q_n^2 - Q_{n+1}^2)/12,\\
J_2 &= I_1J_1/2 - (\delta z_n)^2\rho_n\rho_{n+1}(Q_n^2 - Q_{n+1}^2)/12.
\end{aligned} \tag{48}$$

The layer matrices in the schemes L_1 and L_2 are then

$$\begin{pmatrix} 1 & I_1 \\ -J_1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 - I_2 & I_1 \\ -J_1 & 1 - J_2 \end{pmatrix}. \tag{49}$$

With a symmetrized iteration starting point the corresponding matrices are given by (26) and (29). We shall label calculations using these unimodular matrices by UL_1 and UL_2, respectively.

V. COMPARISON OF THE METHODS, AND CONCLUSIONS

We shall compare the various schemes of numerical calculation of reflection and transmission properties by checking their results against an exactly solvable model stratification, namely, one in which both the density and the speed of sound vary exponentially with depth[14]:

$$\rho(z) = \rho_a e^{(z - z_1)/l}, \quad c = c_a e^{(z - z_1)/L}. \tag{50}$$

The general solution[14] permits a discontinuity in either or both ρ and c at z_1 and/or z_{N+1}, but here we assume continuity of both acoustic variables at both boundaries, so that the lengths l and L are given by

$$l = \Delta z/\log(\rho_b/\rho_a), \quad L = \Delta z/\log(c_b/c_a), \tag{51}$$

where $\Delta z = z_{N+1} - z_1$ is the thickness of the inhomogeneous layer. A (nonabsorbing) layer is characterized by three dimensionless parameters ρ_b/ρ_a, c_b/c_a, and $\omega\Delta z/c_a = 2\pi\Delta z/\lambda_a$, where λ_a is the wavelength in medium a. The exact reflectance and transmittance are easily calculated from the Bessel function solution[14] of Eq. (4), provided the thickness/wavelength parameter $\omega\Delta z/c_a$ is not too large. For large thickness parameter (say greater than ten), the arguments of the Bessel functions become large, the Bessel functions become difficult to calculate from their series expansions, and one must use their asymptotic forms, or general expression[14] for r and t in the short-wave limit. In the same high-frequency, short-wave, or thick-layer limit, large numbers of matrices are required to calculate r and t correctly. In that limit it is better to use the general expressions derived in Ref. 14. In the opposite long-wave (or low-frequency, or thin-layer) limit, very few matrices are needed, and the second-order methods are much more accurate than the first-order methods, for the same number of layer matrices N. We will give the actual errors for seven matrix algorithms in an intermediate case, $\omega\Delta z/c_a = 2$, with $\rho_b = 2\rho_a$, $c_b = (4/3)c_a$. Ten matrices were used in each case. The reflectance was calculated at four angles of incidence, 0° to 45° in 15° steps, over which range it varied from 0.07917 at normal incidence to 0.40968 at 45° [the critical angle for this case is arcsin $(\frac{3}{4}) \approx 48.6°$]. Table I gives the average absolute fractional error in the reflectance, and the average value of det $M - 1$ (proportional to the violation the $R + T = 1$ and $T' = T$ laws), the averages being taken over the four angles of incidence.

We first note that the error could have been reduced in each case by increasing the number of matrices from $N = 10$.

TABLE I. Comparison of seven matrix algorithms, all using 10 matrices. The letters C, L, and U stand for *constant*, *linear*, and *unimodular*. The subscripts denote the order in δz_n to which the matrices are evaluated. The figures are for a stratification in which density and speed vary exponentially (parameters are given in the text).

Method	C_1	L_1	C_2	L_2	C_∞	UL_1	UL_2
Average error(%)	19	19	2.0	0.8	1.3	1.2	0.2
Average of det $M - 1$	2	2	10^{-3}	10^{-3}	0	0	0

The purpose here is to compare the relative errors. At first sight it appears surprising, on comparing Figs. 1 and 2, that the C_1 and L_1 methods give almost the same results. This is because in all the C methods, the (constant) acoustic parameters for each layer were chosen to be the exact values at the middle of the layer. Thus the C_1 and L_1 matrix elements are almost the same [note ρ_n, etc. thus have different meanings in (43) and in (47) and (49)]. Even in second order, the L_2 method is only a factor of 2 or so more accurate, for the same reasons. The improvement in det $M-1$, which should be zero, is however significant. Of the three methods that have det $M=1$ (to within the numerical precision available), C_∞ and UL_1 have comparable accuracy in the reflectance, and are comparable in programming simplicity. However, C_∞ is considerably longer to execute, since the sine and cosine matrix elements take an order of magnitude longer to compute than ones involving only simple arithmetic operations. The unimodular second-order method based on linear variation of acoustic parameter (UL_2) is our preference, followed by UL_1 if very simple matrix elements are required, for example in order to use a small programmable calculator.

We note also that the matrix method can also be used to generate the sound field wavefunctions within the stratification, which are produced in the more usual numeric solution of the differential equation.[16,17] The acoustic pressure field may be found in the same way as the electromagnetic fields, as a by-product of the calculation of the reflection amplitude: See Ref. 10, pp. 247, 248.

In summary, we have formulated the acoustic reflection problem in terms of a product of 2×2 layer matrices, shown that these matrices should be unimodular for energy conservation and a reciprocity theorem to hold, and given two methods that have unimodularity built in and are simple to program.

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