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Localized electromagnetic pulses with azimuthal dependence

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Abstract
We present a family of solutions of the wave equation which are localized in space–time and have azimuthal dependence, thus enlarging the previously known set of solutions. These wavefunctions are used to construct two types of solutions of Maxwell's equations which represent localized electromagnetic pulses. The energy, momentum and angular momentum of these pulses are calculated analytically for simple special cases of the family of wavefunctions. Solutions of the wave equation with $\exp(\mathrm{i}m\phi)$ azimuthal dependence, but otherwise unrestricted, are shown to result in separation of angular momentum into spin and orbital parts for two classes of pulse. However, reversal of the sign of m can produce major changes in the pulse structure.

Keywords: electromagnetic pulses, azimuthal dependence, angular momentum

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Although wavepacket solutions of the Schrödinger equation were found in the early days of quantum mechanics [1, 2], it is only recently that three-dimensionally localized solutions of the wave equation have been explored [3–16]. In most cases the exploration was limited to the scalar wave, but Hellwarth and collaborators [7, 8] constructed an exact localized pulse solution of Maxwell's equations and evaluated its energy. Analytic results for the energy, momentum and angular momentum for this pulse and for related but even simpler pulses were recently presented [14]. It was found that in each case the energy of the pulse was greater than the speed of light times its momentum. It follows that a transformation to the zero-momentum frame of the pulse is possible, and this result has been proven to hold for all space–time localized solutions of Maxwell's equations [15]. One consequence is that localized electromagnetic pulses have an intrinsic angular momentum along the direction of the net momentum of the pulse, unchanged by Lorentz boosts along this direction, and invariant to change of spatial or temporal origin [16].

Most of the localized solutions of the wave equation that have been presented have no azimuthal dependence: in cylindrical polar coordinates (ρ, ϕ and z) the wavefunction is independent of ϕ. The exceptions are the Bessel–Gauss pulses of Overfelt [5], generalized by Kiselev [13]. However, neither author has applied these solutions to electromagnetic waves, and although the formal application is straightforward, the analytic evaluation of the pulse energy, momentum and angular momentum does not appear tractable. In this paper we shall present pulse solutions of the wave equation with azimuthal dependence contained in the factor $\mathrm{e}^{\mathrm{i}m\phi}$, construct solutions of Maxwell's equations from them, and analytically evaluate the energy, momentum and angular momentum for some special cases.

2. Localized solutions of the wave equation with $\mathrm{e}^{\mathrm{i}m\phi}$ azimuthal dependence

Hillion [6] has shown that the wave equation

$$\nabla^2\psi = \partial_t^2\psi \qquad [\partial_t = \partial/\partial(ct)] \tag{1}$$

is solved by the set of functions

$$\psi(\mathbf{r}, t) = \frac{f(s)}{b + \mathrm{i}(z \mp ct)}, \qquad s = \frac{\rho^2}{b + \mathrm{i}(z \mp ct)} - \mathrm{i}(z \pm ct). \tag{2}$$

An alternative route to these solutions has been given in [16]. When $f(s)$ is set equal to $ab\psi_0/(s+a)$ we obtain the Ziolkowski wavefunction [3]

$$\psi_Z = \frac{ab}{\rho^2 + [a - i(z \pm ct)]\,[b + i(z \mp ct)]}\psi_0. \tag{3}$$

(We have put in the factor $ab\psi_0$ so as to normalize the wavefunction to ψ_0 at the space–time origin.) An oscillatory solution with wavenumber k obtained by setting $f(s)$ equal to $ab\psi_0 e^{-ks}/(s+a)$, namely

$$\psi_k(\mathbf{r}, t) = \frac{ab e^{-ks}}{\rho^2 + [a - i(z \pm ct)]\,[b + i(z \mp ct)]}\psi_0 \tag{4}$$

has the plane wave form $e^{ik(z \pm ct)}$ in the region where a and b are large compared to $|z \pm ct|$ and b is large compared to $k\rho^2$. Electromagnetic pulses based on (4) were studied in [16].

We now wish to generate solutions of the wave equation which have azimuthal dependence. We note that

$$\frac{g}{b + i(z \mp ct)}\psi_Z, \qquad g = x, y, x \pm iy \tag{5}$$

are solutions of the wave equation, as are

$$\frac{h}{[b + i(z \mp ct)]^2}\psi_Z, \qquad h = xy, x^2 - y^2, (x \pm iy)^2. \tag{6}$$

The wavefunctions obtained by replacing ψ_Z by $f(s)/[b + i(z \mp ct)]$ in (5) and (6) are also solutions of the wave equation.

We are particularly interested in azimuthal dependence of the form $e^{im\phi}$, where m is a positive or negative integer. We note that $x + iy = \rho e^{i\phi}$, and can verify by differentiation that

$$\left[\frac{\rho}{b + i(z - ct)}\right]^{|m|} e^{im\phi}\frac{f(s)}{b + i(z - ct)},$$
$$s = \frac{\rho^2}{b + i(z - ct)} - i(z + ct) \tag{7}$$

is a solution of the wave equation for arbitrary m and any twice-differentiable function $f(s)$. (Here and henceforth we take the upper sign, as was done in the calculations of [14].)

In quantum mechanics the factor $e^{im\phi}$ would be associated with angular momentum $\hbar m$ about the polar (z) axis. We shall explore the angular momentum properties of pulses derived from the family of solutions (7) in the following sections.

3. TE + iTM pulses with $e^{im\phi}$ azimuthal dependence

Solutions of Maxwell's equations in free space can be obtained from solutions of the wave equation as follows (see for example [17], section 6.4). We set

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\Phi - \partial_t \mathbf{A}. \tag{8}$$

Then provided the scalar potential Φ and all components of the vector potential $\mathbf{A}$ satisfy the wave equation (1), and also satisfy the Lorenz condition

$$\nabla \cdot \mathbf{A} + \partial_t \Phi = 0, \tag{9}$$

the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ will satisfy Maxwell's equations in vacuum.

A transverse-electric (TE) solution is obtained by setting [14]

$$\Phi = 0, \qquad \mathbf{A} = \nabla \times [0, 0, \psi] = \left[\partial_y, -\partial_x, 0\right]\psi. \tag{10}$$

This gives the fields

$$\mathbf{E} = \left[-\partial_y\partial_t, \partial_x\partial_t, 0\right]\psi, \qquad \mathbf{B} = \left[\partial_x\partial_z, \partial_y\partial_t, -\partial_x^2 - \partial_y^2\right]\psi. \tag{11}$$

A transverse-magnetic (TM) solution is obtained by the duality transformation $\mathbf{E} \to \mathbf{B}, \mathbf{B} \to -\mathbf{E}$. The dual of the TE field just given has $\mathbf{E} = \nabla \times \mathbf{A}$, $\mathbf{B} = \partial_t \mathbf{A}$. The combination TE + iTM has

$$\mathbf{B} = \nabla \times \mathbf{A} + i\partial_t \mathbf{A}, \qquad \mathbf{E} = i\mathbf{B}. \tag{12}$$

Monochromatic beams in which the complex fields are related by $\mathbf{E} = \pm i\mathbf{B}$ have energy and momentum densities which do not oscillate in time, in contrast to general beams, in which these oscillate with angular frequency 2ω when the fields oscillate with angular frequency ω. The $\mathbf{E} = \pm i\mathbf{B}$ set of beams, introduced in [18] and further applied in [19], have been called *steady beams*. Pulses clearly cannot be steady in the same sense, but they share with monochromatic beams the property that the energy and momentum densities are given by the simple formulae [15] associated with $\mathbf{E} = \pm i\mathbf{B}$:

$$u = \frac{1}{8\pi}|\mathbf{E}|^2 = \frac{1}{8\pi}|\mathbf{B}|^2 \tag{13}$$

$$c\mathbf{p} = \pm\frac{i}{8\pi}\mathbf{E} \times \mathbf{E}^* = \pm\frac{i}{8\pi}\mathbf{B} \times \mathbf{B}^*. \tag{14}$$

(It is understood in the use of complex fields that the physical fields are either the real or the imaginary parts of $\mathbf{E}$ and $\mathbf{B}$. If for example the real parts are chosen then the energy density is given by $8\pi u = E_r^2 + B_r^2$, which is equal to $E_r^2 + E_i^2$ and hence consistent with (13), since $\mathbf{E}_r + i\mathbf{E}_i = \pm i(\mathbf{B}_r + i\mathbf{B}_i)$ and so $\mathbf{E}_r = \mp\mathbf{B}_i$, $\mathbf{E}_i = \pm\mathbf{B}_r$. Likewise the momentum density is given by $4\pi c\mathbf{p} = \mathbf{E}_r \times \mathbf{B}_r = \pm\frac{i}{2}(\mathbf{E}_r + i\mathbf{E}_i) \times (\mathbf{E}_r - i\mathbf{E}_i) = \pm(\mathbf{E}_r \times \mathbf{E}_i)$ and is thus consistent with (14).)

From (10), (11) and (12) we find that the magnetic field of a TE + iTM pulse is

$$\mathbf{B} = \left[\partial_x\partial_z + i\partial_y\partial_t, \partial_y\partial_z - i\partial_x\partial_t, -\partial_x^2 - \partial_y^2\right]\psi. \tag{15}$$

The energy density is $|\mathbf{B}|^2/8\pi$; on using the wave equation this becomes

$$u = \frac{1}{8\pi}\{|(\partial_x\partial_z + i\partial_y\partial_t)\psi|^2 + |(\partial_y\partial_z - i\partial_x\partial_t)\psi|^2 + |(\partial_z^2 - \partial_t^2)\psi|^2\}. \tag{16}$$

The momentum density is $i\mathbf{B} \times \mathbf{B}^*/8\pi c$, with components

$$p_x = -\frac{1}{4\pi c}\,\mathrm{Im}\left\{\left[(\partial_y\partial_z - i\partial_x\partial_t)\psi\right]\left[\left(\partial_z^2 - \partial_t^2\right)\psi^*\right]\right\}$$
$$p_y = \frac{1}{4\pi c}\,\mathrm{Im}\left\{\left[(\partial_x\partial_z + i\partial_y\partial_t)\psi\right]\left[\left(\partial_z^2 - \partial_t^2\right)\psi^*\right]\right\}$$
$$p_z = -\frac{1}{4\pi c}\,\mathrm{Im}\left\{\left[(\partial_x\partial_z + i\partial_y\partial_t)\psi\right]\left[(\partial_y\partial_z + i\partial_x\partial_t)\psi^*\right]\right\}. \tag{17}$$

In cylindrical polar coordinates $\rho = \sqrt{x^2 + y^2}$, z and ϕ, the x and y derivatives become

$$\partial_x = \cos\phi\,\partial_\rho - \rho^{-1}\sin\phi\,\partial_\phi, \qquad \partial_y = \sin\phi\,\partial_\rho + \rho^{-1}\cos\phi\,\partial_\phi. \tag{18}$$

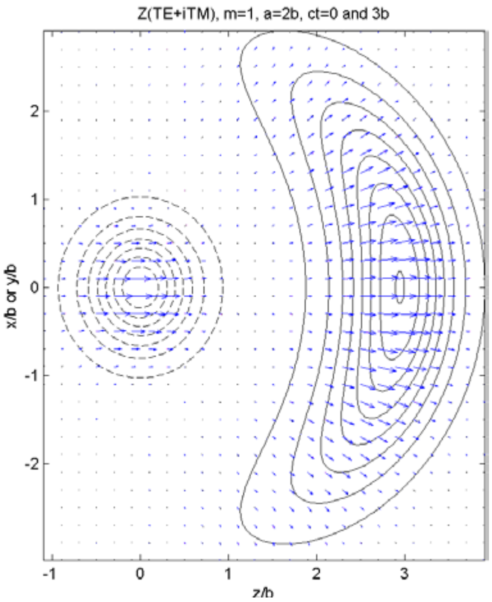

Figure 1. The energy and momentum densities for the $m = 1$ TE + iTM pulse based on the Ziolkowski wavefunction, with $a = 2b$. The energy contours at $t = 0$ are dashed curves; those at $t = 3b/c$ are solid curves. The arrows, indicating the longitudinal and transverse components of the momentum density, are enlarged at $t = 3b/c$ by a factor of 10 for better visibility. At $t = 0$ the pulse is in its focal region, and is most compact. As time increases it spreads.

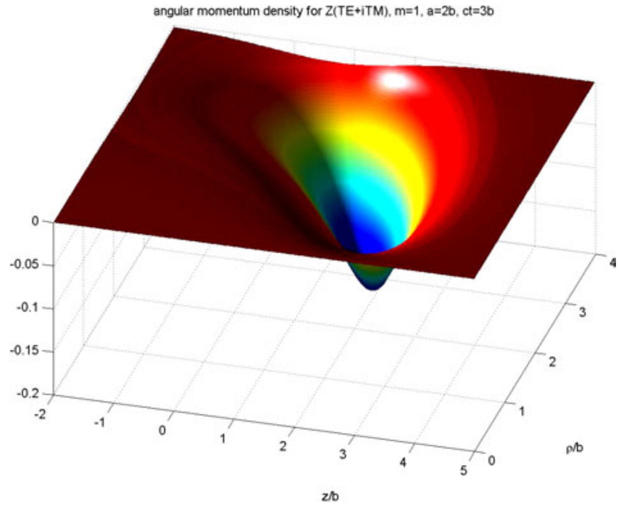

Figure 2. The angular momentum density for the $m = 1$ TE + iTM pulse based on the Ziolkowski wavefunction, with $a = 2b$ (this ratio is used in all the figures), at $ct = 3b$. It is negative-definite, for all positive a and b. The shape of the contours changes as the pulse propagates, but the integral over all space stays constant, at the value given in (26).

In this paper we consider wavefunctions with $e^{im\phi}$ azimuthal dependence ($m = 0, \pm 1, \pm 2 \ldots$). For such wavefunctions $\partial_\phi \psi = im\psi$, and the energy and momentum can be expressed in terms of ρ, z and t derivatives of the wavefunction:

$$u = \frac{1}{8\pi}\{|(\partial_\rho \partial_z - m\rho^{-1}\partial_t)\psi|^2 + |(\partial_\rho \partial_t - m\rho^{-1}\partial_z)\psi|^2 + |(\partial_z^2 - \partial_t^2)\psi|^2\} \tag{19}$$

$$p_z = -\frac{1}{4\pi c}\,\mathrm{Re}\left\{(\partial_\rho\partial_z\psi)\partial_\rho\partial_t\psi^* - \frac{m}{\rho}[(\partial_\rho\partial_z\psi)\partial_z\psi^* + (\partial_\rho\partial_t\psi)\partial_t\psi^*] + \left(\frac{m}{\rho}\right)^2(\partial_z\psi)\partial_t\psi^*\right\}. \tag{20}$$

Instead of p_x and p_y we shall work with the radial and azimuthal components of the momentum density,

$$p_\rho = p_x\cos\phi + p_y\sin\phi, \qquad p_\phi = -p_x\sin\phi + p_y\cos\phi. \tag{21}$$

For the TE + iTM pulse with $e^{im\phi}$ azimuthal dependence, these are

$$p_\rho = \frac{1}{4\pi c}\,\mathrm{Re}\left\{\left[\left(\partial_\rho - \frac{m}{\rho}\right)\partial_t\psi\right]\left(\partial_z^2 - \partial_t^2\right)\psi^*\right\} \tag{22}$$

$$p_\phi = \frac{1}{4\pi c}\,\mathrm{Im}\left\{\left[\left(\partial_\rho\partial_z - \frac{m}{\rho}\partial_t\right)\psi\right]\left(\partial_z^2 - \partial_t^2\right)\psi^*\right\}. \tag{23}$$

We note that all of u, p_z, p_ρ and p_ϕ are independent of the azimuthal angle ϕ, and that u and p_z have terms in m^0, m and m^2, while p_ρ and p_ϕ have only m^0 and m terms.

The above results are for any pulse solution of the wave equation which has $e^{im\phi}$ azimuthal dependence. In [14] we evaluated the total energy, momentum and angular momentum of a TE + iTM pulse constructed from the $m = 0$ Ziolkowski wavefunction ψ_Z given in (3):

$$U = \frac{\pi}{8}\frac{a+b}{ab}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{a-b}{ab}\psi_0^2, \qquad J_z = 0. \tag{24}$$

(The result that $J_z = 0$ for any TE + iTM pulse with wavefunction independent of ϕ is proved in [16].) For the wavefunctions (based on ψ_Z with the upper sign taken in (3))

$$\psi_\pm = \frac{x \pm iy}{b + i(z - ct)}\psi_Z = \frac{\rho e^{\pm i\phi}}{b + i(z - ct)}\psi_Z \tag{25}$$

we find, by the methods described in [14],

$$U = \frac{\pi}{8}\frac{3a+b}{b^2}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{3a-b}{b^2}\psi_0^2, \qquad cJ_z = \mp\frac{\pi}{4}\frac{a}{b}\psi_0^2. \tag{26}$$

The energy, momentum and angular momentum densities have different functional forms for the ψ_+ and ψ_- wavefunctions, but have the same total energy and momentum, and the total angular momenta differ in sign only. We note that the Lorentz boost required for transformation to the zero-momentum frame, which was $\beta = (a - b)/(a + b)$ for ψ_Z, now becomes $\beta = (3a - b)/(3a + b)$. The sign of J_z is perhaps surprizing, since $e^{+i\phi}$ azimuthal dependence would in quantum mechanics be expected to give a positive J_z.

The energy and momentum densities for the ψ_+ TE + iTM pulse are shown in figure 1, and the angular momentum density

$$j_z = xp_y - yp_x = \rho p_\phi \tag{27}$$

is shown in figure 2. All of u, $\mathbf{p}$ and j_z depend on time as well as on position. However, the total energy, momentum and angular momentum

$$U = \int d^3r\, u(\mathbf{r}, t), \qquad \mathbf{P} = \int d^3r\, \mathbf{p}(\mathbf{r}, t), \qquad \mathbf{J} = \int d^3r\, \mathbf{r} \times \mathbf{p} \tag{28}$$

have been shown to be constants [14, 16], and can thus be evaluated at any time. Figure 1 shows the ψ_+ densities at $t = 0$ and $3b/c$. Figure 2 shows the z-component of the angular momentum density j_z, which is negative-definite. The change $m \to -m$ reverses the sign of J_z, as may be expected. This result follows from the fact that (23) may be written as

$$p_\phi = p_\phi^{(0)} - \frac{m}{4\pi c\rho} \operatorname{Im}\left\{(\partial_t \psi)\left(\partial_z^2 - \partial_t^2\right)\psi^*\right\} \tag{29}$$

where $\int d^3r\, \rho p_\phi^{(0)}$ is zero by the arguments given in section 5 of [16].

4. 'CP' pulses with $e^{im\phi}$ azimuthal dependence

The 'circularly polarized' ('CP') pulse was introduced in section 3 of [16]. The inverted commas are used for two reasons:

(i) pulses cannot be monochromatic and thus the electric and magnetic field vectors do not periodically orbit elliptical paths at a given point in space as they do in monochromatic beams (these elliptical paths define the polarization—see for example [20]), and

(ii) even for nearly monochromatic pulses, which locally can resemble a monochromatic beam for a short time, perfect circular polarization in a fixed plane is not possible (theorem 2.3 of section 2 of [20]).

Nevertheless, the 'CP' pulses defined in [14] and [16] by

$$\mathbf{A} = \nabla \times [i\psi, \psi, 0], \qquad \mathbf{B} = \nabla \times \mathbf{A} + i\partial_t \mathbf{A}, \qquad \mathbf{E} = i\mathbf{B} \tag{30}$$

have twist in the fields even when ψ is independent of the azimuthal angle, and for nearly monochromatic pulses will have a central region in which the fields are close to circularly polarized (the polarization of the related 'CP' beam is discussed in section 4 of [20]).

'CP' pulses defined by (30) have energy and momentum densities given by (13) and (14). We shall evaluate these for general ψ, and then specialize for solutions of the wave equation which have the azimuthal dependence $e^{im\phi}$, as we did for TE + iTM pulses. The magnetic field has, from (30), the components

$$\begin{aligned} B_x &= (\partial_x - i\partial_y)\partial_y\psi - i(\partial_z + \partial_t)\partial_z\psi \\ B_y &= -(\partial_x - i\partial_y)\partial_x\psi - (\partial_z + \partial_t)\partial_z\psi \\ B_z &= i(\partial_x - i\partial_y)(\partial_z + \partial_t)\psi. \end{aligned} \tag{31}$$

The energy density is $(|B_x|^2 + |B_y|^2 + |B_z^2|)/8\pi$. The momentum density is $i\mathbf{B} \times \mathbf{B}^*/8\pi c$; its components are

$$\begin{aligned} p_x &= -\frac{1}{4\pi c}\operatorname{Re}\{[(\partial_x - i\partial_y)\partial_x\psi + (\partial_z + \partial_t)\partial_z\psi] \\ &\quad \times (\partial_x + i\partial_y)(\partial_z + \partial_t)\psi^*\} \\ p_y &= -\frac{1}{4\pi c}\operatorname{Re}\{[(\partial_x - i\partial_y)\partial_y\psi - i(\partial_z + \partial_t)\partial_z\psi] \\ &\quad \times (\partial_x + i\partial_y)(\partial_z + \partial_t)\psi^*\} \\ p_z &= \frac{1}{4\pi c}\operatorname{Im}\{[(\partial_x - i\partial_y)\partial_y\psi - i(\partial_z + \partial_t)\partial_z\psi] \\ &\quad \times [(\partial_x + i\partial_y)\partial_x\psi^* + (\partial_z + \partial_t)\partial_z\psi^*]\}. \end{aligned} \tag{32}$$

We now consider wavefunctions which have $e^{im\phi}$ azimuthal dependence, so that $\partial_\phi\psi = im\psi$. We shall thus use (18) in the form

$$\begin{aligned} \partial_x\psi &= \cos\phi\,\partial_\rho\psi - \frac{im}{\rho}\sin\phi\,\psi, \\ \partial_y\psi &= \sin\phi\,\partial_\rho\psi + \frac{im}{\rho}\cos\phi\,\psi. \end{aligned} \tag{33}$$

Also from (18), without restriction,

$$\partial_x \pm i\partial_y = e^{\pm i\phi}\left(\partial_\rho \pm \frac{i}{\rho}\partial_\phi\right). \tag{34}$$

We then find, for $e^{im\phi}$ azimuthal dependence, that the energy density u and also the longitudinal, radial and azimuthal components of the momentum density are all independent of the azimuthal angle ϕ. As for the TE + iTM pulse, u and p_z are of second degree in m, while p_ρ and p_ϕ are of first degree in m. The terms of zero degree in m have been given in equations (11) and (14) of [16], so we shall just give to terms of first and second degree in m:

$$\begin{aligned} u^{(1)} = \frac{m}{4\pi}[&\rho^{-1}\operatorname{Re}\{(\partial_z + \partial_t)\psi^*(\partial_z + \partial_t)\partial_\rho\psi \\ &- (\partial_\rho\psi^*)(\partial_z + \partial_t)\partial_z\psi\} - \rho^{-2}[\operatorname{Re}(\psi^*\partial_\rho^2\psi) + |\partial_\rho\psi|^2] \\ &+ \rho^{-3}\operatorname{Re}(\psi^*\partial_\rho\psi)] \end{aligned} \tag{35}$$

$$\begin{aligned} u^{(2)} &= \frac{m^2}{8\pi}[\rho^{-2}(|\partial_\rho\psi|^2 + |(\partial_z + \partial_t)\psi|^2) \\ &\quad - 2\rho^{-3}\operatorname{Re}(\psi^*\partial_\rho\psi) + 2\rho^{-4}|\psi|^2] \\ p_z^{(1)} &= \frac{m}{4\pi c}[\rho^{-1}\operatorname{Re}\{(\partial_\rho\psi^*)(\partial_\rho^2 + \partial_z^2 + \partial_z\partial_t)\psi\} \\ &\quad - \rho^{-2}\operatorname{Re}\{\psi^*(\partial_\rho^2 - \rho^{-1}\partial_\rho)\psi\}] \end{aligned} \tag{36}$$

$$\begin{aligned} p_z^{(2)} &= \frac{m^2}{4\pi c}[-\rho^{-3}\operatorname{Re}(\psi^*\partial_\rho\psi) + \rho^{-4}|\psi|^2] \\ p_\rho^{(1)} &= \frac{m}{4\pi c}[-\rho^{-1}\operatorname{Re}\{(\partial_\rho^2 + \partial_z^2 + \partial_z\partial_t)\psi^*(\partial_z + \partial_t)\psi\} \\ &\quad + \rho^{-2}\operatorname{Re}\{\psi^*(\partial_z + \partial_t)\partial_\rho\psi\}] \end{aligned} \tag{37}$$

$$\begin{aligned} p_\phi^{(1)} &= \frac{m}{4\pi c}[\rho^{-1}\operatorname{Im}\{(\partial_z + \partial_t)\partial_z\psi^*(\partial_z + \partial_t)\psi \\ &\quad - (\partial_\rho\psi^*)(\partial_z + \partial_t)\partial_\rho\psi\} \\ &\quad + \rho^{-2}\operatorname{Im}\{\psi^*(\partial_z + \partial_t)\partial_\rho\psi + (\partial_\rho\psi^*)(\partial_z + \partial_t)\psi\}]. \end{aligned} \tag{38}$$

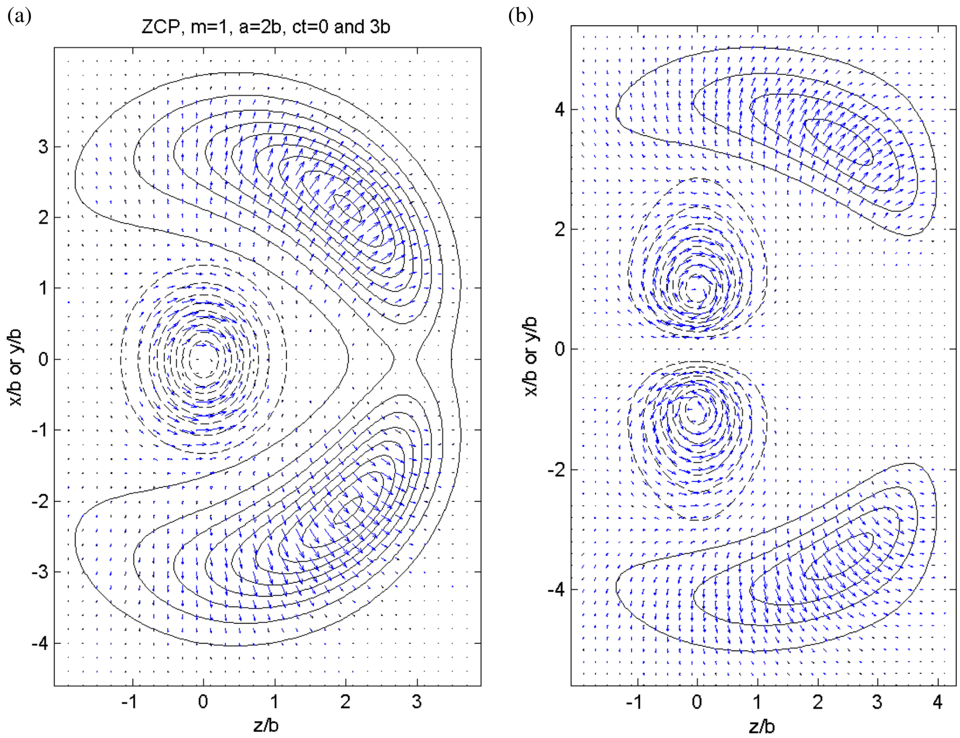

Figure 3. (a) The energy and momentum densities for the $m = 1$ 'CP' pulse based on the Ziolkowski wavefunction, with $a = 2b$. The energy contours at $t = 0$ are dashed curves; those at $t = 3b/c$ are solid curves. The arrows, indicating the longitudinal and transverse components of the momentum density, are enlarged at $t = 3b/c$ for better visibility. The pulse is hollow in momentum and energy, propagating forward and outward as an annulus. (b) As for (a), but now for the $m = -1$ 'CP' pulse, and at $t = 0$ and $4b/c$. Note the vortex ring structure, most marked at $t = 0$, where momentum backflow is strong. Note also the differences between this diagram and that of (a): it is remarkable that reversal of the sign of m produces qualitative changes in the pulse.

The total energy, momentum and angular momentum of the 'CP' pulse resulting from the Ziolkowski wavefunction (3) were evaluated in [14]:

$$U = \frac{\pi}{8}\frac{a+3b}{a^2}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{a-3b}{a^2}\psi_0^2, \qquad cJ_z = \frac{\pi}{4}\frac{b}{a}\psi_0^2. \tag{39}$$

The results for the $m = \pm 1$ solutions of the wave equation given in (25) are

$$U = \frac{\pi}{8}\frac{a+b}{ab}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{a-b}{ab}\psi_0^2, \qquad J_z = \begin{cases} 0 & (m=1) \\ \frac{\pi}{4}\psi_0^2 & (m=-1). \end{cases} \tag{40}$$

We note that the angular momentum in (39) results from twist in the fields: the wavefunction ψ_Z has no azimuthal dependence. When azimuthal dependence is introduced, it can cancel or enhance the angular momentum contained in the azimuthally-independent part.

Figure 3(a) shows the energy density and the longitudinal and radial momentum densities of the ψ_+ 'CP' pulse, and figure 4 shows the angular momentum density $j_z = \rho p_\phi$ for the $m = 1$ pulse. Figures 3(b) and 5 show the corresponding results for the $m = -1$ pulse. There are positive and negative parts of j_z for the $m = 1$ pulse, which exactly cancel in the integral $J_z = \int d^3r\, j_z$, while the $m = -1$ pulse has a positive-definite angular momentum density.

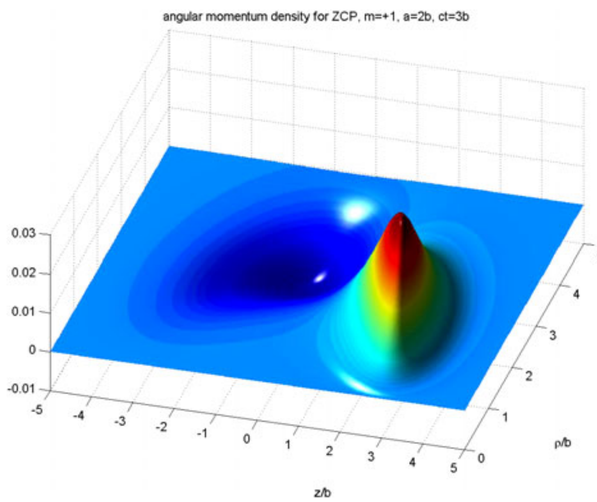

Figure 4. The angular momentum density for the $m = 1$ 'CP' pulse based on the Ziolkowski wavefunction with $a = 2b$, at $ct = 3b$. It has positive and negative regions, whose shape changes as the pulse propagates, but the integral over all space is always zero.

5. Discussion

We have shown how the Hillion class of axially symmetric solutions of the wave equation can be enlarged to include azimuthal dependence. When the simplest of these wavefunctions are applied to generate electromagnetic pulses of the TE + iTM and 'CP' types, the energy, momentum and

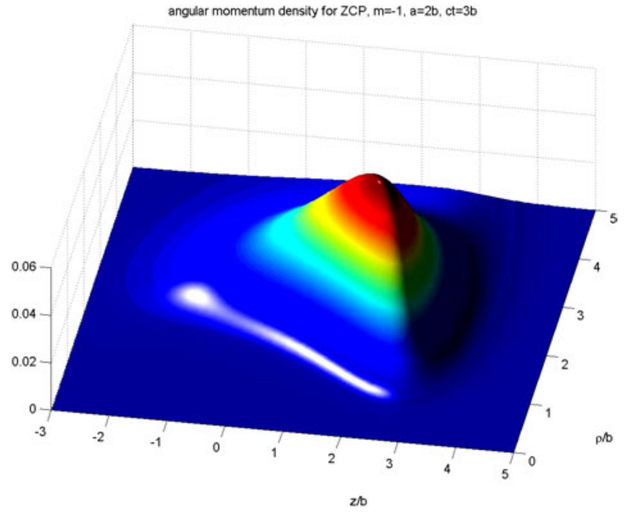

Figure 5. As for figure 4, but now for the $m = -1$ 'CP' pulse, at $ct = 3b$. In contrast to the $m = 1$ case, the angular momentum density is positive-definite. The integral over all space is constant in time, the value being given by (40).

angular momentum all change. In the TE + iTM case the azimuthally-dependent wavefunction gave non-zero angular momentum (as expected), but of opposite helicity to that expected. In the 'CP' cases the angular momentum is non-zero for wavefunctions independent of the azimuthal angle, and we found that the simplest azimuthally-dependent wavefunctions, with $m = \pm 1$, respectively cancel and enhance the angular momentum of the pulse. The fact that the cancellation is exact when $m = +1$ could be interpreted as being the cancellation of spin and orbital angular momentum, with values +1 and -1 respectively, since a vector field has spin 1 (see for example [21]), and $m = \pm 1$ corresponds to orbital angular momentum 1 in scalar particle wavemechanics.

The practice of separating the angular momentum content of light beams into orbital and spin components is well established: see problems 7.19–7.21 in [17] and the reprint collection [22]. Bartlett [23] has considered a class of beams in which the fields are constructed from wavefunctions composed of products of an amplitude, a plane wave, and $e^{im\phi}$ times a Bessel function of order m. For these beams the angular momentum flux separates into spin and orbital parts, and the orbital part is proportional to m. However, the beam invariants arising out of the conservation of angular momentum [24] do not appear to separate into spin and orbital parts, in general.

The expressions for the angular momentum of the classical electromagnetic pulses considered here can likewise be considered to consist of a spin part (independent of m) and an orbital part (proportional to m), at least in the case of wavefunctions with $e^{im\phi}$ azimuthal dependence. However, as we have seen, reversal of the sign of m can produce major changes in the pulse structure.

When the wavefunctions have more complicated azimuthal dependence, the separation into spin and orbital parts is not obvious, for pulses or for beams.

References

[1] Kennard E H 1927 *Z. Phys.* **44** 326–52
[2] Darwin C G 1928 *Proc. R. Soc.* A **117** 258–93
[3] Ziolkowski R W 1985 *J. Math. Phys.* **26** 861–3
Ziolkowski R W 1989 *Phys. Rev.* A **39** 2005–33
[4] Shaarawi A M, Besieris I M and Ziolkowski R W 1989 *J. Appl. Phys.* **65** 805–11
[5] Overfelt P L 1991 *Phys. Rev.* A **44** 3941–7
[6] Hillion P 1993 *Acta Appl. Math.* **30** 35–45
[7] Hellwarth R W and Nouchi P 1996 *Phys. Rev.* E **54** 889–95
[8] Feng S, Winful H G and Hellwarth R W 1999 *Phys. Rev.* E **59** 4630–49
[9] Borzdov G N 2000 *Phys. Rev.* E **61** 4462–78
Borzdov G N 2001 *Phys. Rev.* E **63** 036606(10)
[10] Reivelt K and Saari P 2000 *J. Opt. Soc. Am.* A **17** 1785–90
Reivelt K and Saari P 2002 *Phys. Rev.* E **66** 056611(9)
[11] Kiselev A P and Perel M V 2000 *J. Math. Phys.* **41** 1934–55
[12] Saari P 2001 *Opt. Express* **8** 590–8
[13] Kiselev A P 2003 *J. Phys. A: Math. Gen.* **36** L345–9
[14] Lekner J 2003 *J. Opt. A: Pure Appl. Opt.* **5** L15–8
[15] Lekner J 2004 *J. Opt. A: Pure Appl. Opt.* **6** 146–7
[16] Lekner J 2004 *J. Opt. A: Pure Appl. Opt.* **6** S128–33
[17] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley)
[18] Lekner J 2001 *J. Opt. A: Pure Appl. Opt.* **3** 407–12
[19] Lekner J 2002 *J. Opt. A: Pure Appl. Opt.* **4** 491–9
[20] Lekner J 2003 *J. Opt. A: Pure Appl. Opt.* **5** 6–14
[21] Weissbluth M 1978 *Atoms and Molecules* (New York: Academic)
[22] Allen L, Barnett S M and Padgett M J (ed) 2003 *Optical Angular Momentum* (Bristol: Institute of Physics Publishing)
[23] Bartlett S M 2004 *J. Opt. B: Quantum Semiclass. Opt.* **4** S1–10
[24] Lekner J 2004 *J. Opt. A: Pure Appl. Opt.* **6** 204–9