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Invariants of three types of generalized Bessel beams

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Abstract
Barnett and Allen (1994 *Opt. Commun.* **110** 670–8) used a superposition of Bessel beams to theoretically construct a monochromatic light beam with finite energy, momentum and angular momentum contents per unit length. The same superposition of Bessel functions gives solutions of the Helmholtz equation with exp($\mathrm{i}m\phi$) azimuthal dependence, from which exact solutions of Maxwell's equations are obtained for three types of beam: TM (or TE), TE + iTM and 'CP' ('circularly polarized'). The invariants of these three types of beam are calculated and compared. One remarkable fact is that the energy and angular momentum contents per unit length are invariants for all the beams considered, whereas these quantities are in general dependent on the longitudinal beam coordinate.

Keywords: Bessel beams, invariants, laser beams

1. Introduction

Barnett and Allen [1] have used the fact that the Helmholtz equation is separable in cylindrical coordinates ρ, z, ϕ to construct a light beam in which the electric field has the form

$$\mathbf{E}(\rho, z, \phi) = \int_0^k \mathrm{d}\kappa\, f(\kappa)\left\{(\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}})F_m + \frac{\kappa}{2q}\hat{\mathbf{z}}[(\mathrm{i}\alpha - \beta)F_{m-1} - (\mathrm{i}\alpha + \beta)F_{m+1}]\right\}. \quad (1)$$

Here and henceforth we use as shorthand the notations

$$q = \sqrt{k^2 - \kappa^2}, \qquad F_m(\rho, \phi, z) = \mathrm{e}^{\mathrm{i}qz}J_m(\kappa\rho)\mathrm{e}^{\mathrm{i}m\phi}. \quad (2)$$

Barnett and Allen found expressions for the energy per unit length, and for the linear and angular momenta per unit length. Remarkably, all three of these quantities were found to be independent of z (the longitudinal coordinate of the beam), whereas only the linear momentum content per unit length is an invariant in general [2]. Invariants (that is quantities which are the same anywhere along the length of the beam) have been shown to follow from conservation laws; there are seven universal invariants, corresponding to the conservation of energy, momentum, and angular momentum [2]. These universal invariants all have the form of an integral of a flux density over a section of the beam. For example, conservation of energy is expressed in terms of real fields $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ by [3]

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = 0, \qquad u = \frac{1}{8\pi}(E^2 + B^2), \qquad \mathbf{S} = c^2\mathbf{p} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \quad (3)$$

(u is the energy density, $\mathbf{S}$ the energy flux density, $\mathbf{p}$ the momentum density of the electromagnetic fields which constitute the light beam). It follows from (3) that

$$P_z' = \int \mathrm{d}^2r\, \bar{p}_z \quad \text{is independent of } z \quad (4)$$

(a bar denotes cycle averaging: the beam is assumed to be monochromatic). Here and throughout this paper, $\int \mathrm{d}^2r$ denotes $\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y$, i.e. integration over the transverse coordinates of the beam, at fixed z.

The quantity P_z' can be interpreted as the longitudinal momentum content per unit length of the beam: $\mathrm{d}P_z = P_z'\,\mathrm{d}z$ is the momentum content in a slice of thickness $\mathrm{d}z$. Note that the momentum content per unit length of the beam is an invariant because of *energy* conservation; the momentum enters equation (3) through the relation $\mathbf{S} = c^2\mathbf{p}$ between the energy flux density and the momentum density. The energy content per unit length, and the z-component of angular momentum content per unit length,

$$U' = \int \mathrm{d}^2 r\, \bar{u}, \qquad J_z' = \int \mathrm{d}^2 r\, \mathbf{r} \times \bar{\mathbf{p}} \tag{5}$$

are not universal invariants, but they are independent of z for the Barnett and Allen beam [1].

We shall explore three types of light beams based on the Barnett and Allen type of wavefunction. These beams are designated TM, TE + iTM and 'CP', with TM denoting transverse magnetic, TE transverse electric, and 'CP' circularly polarized (the quotation marks are explained in section 4). We shall evaluate the universal invariants and special invariants for all three types, and show that all of these beams have a greater number of invariants than is expected.

2. Construction of fields from solutions of the scalar Helmholtz equation

Barnett and Allen [1] considered monochromatic beams in which the x and y components of the electric field have $e^{im\phi}$ dependence, given in equation (1). The associated magnetic field components have been written out in full by Barnett [4]. We shall use the fact that the F_m of equation (2) satisfy the Helmholtz equation $(\nabla^2 + k^2)F_m = 0$ to construct the most general monochromatic beam based on the F_m, and then consider three special classes of beam.

In the Lorenz gauge, and with all time dependence in the factor $e^{-i\omega t}$, the complex electric and magnetic fields can be obtained as spatial derivatives of the complex vector potential $\mathbf{A}(\mathbf{r})$, each component of which must satisfy the Helmholtz equation with $k = \omega/c$ [3, 5]:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}), \qquad \mathbf{E}(\mathbf{r}) = \frac{i}{k}[\nabla(\nabla \cdot \mathbf{A}(\mathbf{r})) + k^2 \mathbf{A}(\mathbf{r})] \tag{6}$$

(the physical fields are then obtained as the real or imaginary parts of $\mathbf{B}(\mathbf{r})e^{-i\omega t}$, $\mathbf{E}(\mathbf{r})e^{-i\omega t}$). In the Barnett and Allen case, $\mathbf{E}$ is given in equation (1), and $\nabla \cdot \mathbf{E} = 0$, so

$$\mathbf{A}(\mathbf{r}) = (ik)^{-1}\mathbf{E}(\mathbf{r}) \tag{7}$$

satisfies the second equality in (6). Since the F_m satisfy the Helmholtz equation, so do all the components of $\mathbf{E}$ and of $\mathbf{A}$.

Let ψ_1, ψ_2 and ψ_3 all be solutions of $(\nabla^2 + k^2)\psi = 0$. Then the most general monochromatic beam is formed from (6) with

$$\mathbf{A} = [\psi_1, \psi_2, \psi_3]. \tag{8}$$

In this paper we shall examine three other types of beam: the TM (or TE), TE + iTM and 'CP' beams. These have vector potentials based on a single solution ψ of the Helmholtz equation [5–7]:

$$\text{TM:} \quad \mathbf{A}_{\text{TM}} = [0, 0, \psi], \qquad \mathbf{B}_{\text{TM}} = [\partial_y, -\partial_x, 0]\psi \tag{9}$$

$$\text{TE:} \quad \mathbf{A}_{\text{TE}} = (ik)^{-1}\nabla \times \mathbf{A}_{\text{TM}} = (ik)^{-1}\mathbf{B}_{\text{TM}},$$
$$\mathbf{E}_{\text{TE}} = [\partial_y, -\partial_x, 0]\psi \tag{10}$$

$$\text{TE + iTM:} \quad \mathbf{A} = k^{-1}[\partial_y, -\partial_x, k]\psi,$$
$$\mathbf{B} = k^{-1}[\partial_x\partial_z + k\partial_y, \partial_y\partial_z - k\partial_x, \partial_z^2 + k^2]\psi \tag{11}$$

$$\text{'CP':} \quad \mathbf{A} = \mathbf{A}_1 + k^{-1}\nabla \times \mathbf{A}_1, \qquad \mathbf{A}_1 = [-i, 1, 0]\psi. \tag{12}$$

The TM beam has the simplest vector potential with only one non-zero component: $\mathbf{A}$ is directed along the direction of beam propagation. The TE beam is the dual [3] of the TM beam, obtained by setting $\mathbf{E} \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}$. The TE + iTM and 'CP' beams have the important property that $\mathbf{E} = i\mathbf{B}$. Beams with this property (or with $\mathbf{E} = -i\mathbf{B}$) have been termed *steady beams* [5, 6]: their energy and momentum densities are independent of time, in contrast to the oscillation at angular frequency 2ω which occurs in other beams. The energy and momentum densities of steady beams with $\mathbf{E} = i\mathbf{B}$ are simply expressed in terms of the complex fields $\mathbf{E}(\mathbf{r})$ or $\mathbf{B}(\mathbf{r})$:

$$u = \frac{1}{8\pi}|\mathbf{E}|^2 = \frac{1}{8\pi}|\mathbf{B}|^2, \qquad c\mathbf{p} = \frac{i}{8\pi}\mathbf{E} \times \mathbf{E}^* = \frac{i}{8\pi}\mathbf{B} \times \mathbf{B}^* \tag{13}$$

(for steady beams with $\mathbf{E} = -i\mathbf{B}$ the sign of $\mathbf{p}$ in (13) is reversed). Likewise all elements of the Maxwell stress tensor are time-independent when $\mathbf{E} = \pm i\mathbf{B}$, so no cycle-averaging is required in the calculation of the beam invariants [2].

In this paper we shall use the fact that $F_m(\mathbf{r})$ defined in (2) satisfies the Helmholtz equation, and set

$$\psi_m(\mathbf{r}) = e^{im\phi} \int_0^k d\kappa\, f(\kappa) e^{iqz} J_m(\kappa\rho) \qquad (q^2 + \kappa^2 = k^2). \tag{14}$$

Our main interest will be in calculating the invariants of the various types of beams when these are based on the wavefunction (14), with m zero or a positive or negative integer. The only constraint on the function $f(\kappa)$ is that certain integrals of $|f(\kappa)|^2$ over κ should exist.

3. Invariants of TM and TE beams

The TM and TE beams are duals, obtained from each other by the duality transformation $\mathbf{E} \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}$. Thus the energy density $u = (8\pi)^{-1}(E^2 + B^2)$ and momentum density $\mathbf{p} = (4\pi c)^{-1}\mathbf{E} \times \mathbf{B}$ are the same for the TM and TE beams based on the same wavefunction. Likewise the elements of the Maxwell stress tensor are unchanged under the duality transformation. Hence all the seven invariants [2] arising from the conservation laws are the same for the TM and TE beams. We shall work with the TM fields, defined in (9).

The wavefunction ψ in (14) is expressed in cylindrical coordinates ρ, ϕ, z. In terms of these coordinates the derivatives that we need (grad, div and curl) are given by

$$\nabla\psi = (\hat{\boldsymbol{\rho}}\partial_\rho + \hat{\boldsymbol{\varphi}}\rho^{-1}\partial_\phi + \hat{\mathbf{z}}\partial_z)\psi \tag{15}$$

$$\nabla \cdot \mathbf{A} = (\partial_\rho + \rho^{-1})A_\rho + \rho^{-1}\partial_\phi A_\phi + \partial_z A_z \tag{16}$$

$$\nabla \times \mathbf{A} = \hat{\boldsymbol{\rho}}(\rho^{-1}\partial_\phi A_z - \partial_z A_\phi) + \hat{\boldsymbol{\varphi}}(\partial_z A_\rho - \partial_\rho A_z)$$
$$+ \hat{\mathbf{z}}[(\partial_\rho + \rho^{-1})A_\phi - \rho^{-1}\partial_\phi A_\rho]. \tag{17}$$

For the TM beam defined in (9) the complex fields are

$$\mathbf{B}(\mathbf{r}) = (\hat{\boldsymbol{\rho}}\rho^{-1}\partial_\phi - \hat{\boldsymbol{\varphi}}\partial_\rho)\psi \tag{18}$$

$$\mathbf{E}(\mathbf{r}) = \frac{i}{k}[\hat{\boldsymbol{\rho}}\partial_\rho\partial_z + \hat{\boldsymbol{\varphi}}\rho^{-1}\partial_\phi\partial_z + \hat{\mathbf{z}}(\partial_z^2 + k^2)]\psi. \tag{19}$$

We wish to calculate the non-zero beam invariants, and also the quantities which are not invariant in general, like the energy content per unit length of the beam U', defined in (5). The time-averaged energy density is [3]

$$\bar{u} = \frac{1}{16\pi}[\mathbf{E} \cdot \mathbf{E}^* + \mathbf{B} \cdot \mathbf{B}^*]. \tag{20}$$

In calculating

$$U' = \int \mathrm{d}^2 r\, \bar{u} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \bar{u} = \int_0^{\infty} \mathrm{d}\rho\, \rho \int_0^{2\pi} \mathrm{d}\phi\, \bar{u} \tag{21}$$

from (18) to (20) with the wavefunction (14) we always have ϕ-independent terms, since $\partial_\phi \psi = \mathrm{i}m\psi$. Thus the integration over ϕ just gives a factor of 2π. We are left with the integration over ρ, resulting in terms of the type

$$I^{\ell}_{nn'} = \int_0^{\infty} \mathrm{d}\rho\, \rho^{\ell} \int_0^k \mathrm{d}\kappa\, f(\kappa) J_n(\kappa\rho) \times \int_0^k \mathrm{d}\kappa'\, f^*(\kappa') J_{n'}(\kappa'\rho) \mathrm{e}^{\mathrm{i}(q-q')z}. \tag{22}$$

The value of ℓ is 1, 0 or -1. The values of (n, n') are (m, m), $(m + 1, m)$ and $(m, m + 1)$, since differentiation with respect to ρ in the evaluation of the fields gives us [8, section 2.12]

$$\partial_\rho J_m(\kappa\rho) = \rho^{-1} m J_m(\kappa\rho) - \kappa J_{m+1}(\kappa\rho). \tag{23}$$

The mixed terms with $n \neq n'$ cancel exactly in U' (see appendix A), so we are left with an $n = n' = m$ term. As noted by Barnett and Allen [1], this simplifies with the aid of Hankel's integral formula [8, section 14.4], [9, section 8.3], [10, section 4.9]. For the problem in hand we need it in the form

$$\int_0^{\infty} \mathrm{d}\rho\, \rho \int_0^k \mathrm{d}\kappa\, \kappa f(\kappa) J_m(\kappa\rho) J_m(\kappa'\rho) = f(\kappa') \quad \text{for } 0 < \kappa' < k. \tag{24}$$

The integral $\int_0^k \mathrm{d}\kappa\, \sqrt{\kappa} f(\kappa)$ must exist and be absolutely convergent. If f is discontinuous, $f(\kappa')$ on the right-hand side of (24) is to be replaced by $\frac{1}{2}[f(\kappa' - 0) + f(\kappa' + 0)]$. We can rewrite (24) by using Dirac's delta function (see appendix B for a discussion of this and of related singular integrals):

$$\int_0^{\infty} \mathrm{d}\rho\, \rho J_m(\kappa\rho) J_m(\kappa'\rho) = \kappa^{-1} \delta(\kappa - \kappa'). \tag{25}$$

The effect of (24), (25) is to reduce the three integrations (over ρ, κ and κ') to one. We find, for the TM beam,

$$U' = \tfrac{1}{4} \int_0^k \mathrm{d}\kappa\, \kappa |f(\kappa)|^2. \tag{26}$$

This result is remarkable in two ways: U' is independent of z, despite not being an invariant for an arbitrary TM beam (an example of a z-dependent U' is given in [5] and [2]). Also, U' is independent of m, i.e. it takes the same value for the infinity of TM beams based on the wavefunctions ψ_m of (14), with m an arbitrary integer or zero (the integer value is not dictated by (23) or (24), which hold for non-integral m, but by the requirement that ψ_m be a periodic function of ϕ with period 2π).

The cycle-averaged momentum content per unit length of the beam is always independent of z [2]. It is

$$P'_z = \int \mathrm{d}^2 r\, \bar{p}_z = \int_0^{\infty} \mathrm{d}\rho\, \rho \int_0^{2\pi} \mathrm{d}\phi\, \bar{p}_z. \tag{27}$$

The cycle-averaged momentum density is given by [3]

$$\bar{\mathbf{p}} = \frac{1}{16\pi c} (\mathbf{E} \times \mathbf{B}^* + \mathbf{E}^* \times \mathbf{B}). \tag{28}$$

It is convenient to work in terms of the radial and azimuthal components of the field, given in (18) and (19). In terms of these we have

$$\bar{p}_z = \frac{1}{8\pi c} \mathrm{Re}(E_\rho B^*_\phi - E_\phi B^*_\rho). \tag{29}$$

This is independent of ϕ when ψ has $\mathrm{e}^{\mathrm{i}m\phi}$ azimuthal dependence. We find that the integrand of $\bar{p}_z$ consists of $(16\pi ck)^{-1}(q + q')\mathrm{e}^{\mathrm{i}(q-q')z}$ times the following terms:

$$\kappa\kappa' J_{m+1}(\kappa\rho) J_{m+1}(\kappa'\rho) - m\rho^{-1}\{\kappa J_{m+1}(\kappa\rho) J_m(\kappa'\rho) + \kappa' J_m(\kappa\rho) J_{m+1}(\kappa'\rho)\} + 2m^2\rho^{-2} J_m(\kappa\rho) J_m(\kappa'\rho). \tag{30}$$

The first term contributes $(4ck)^{-1} \int_0^k \mathrm{d}\kappa\, \kappa q |f(\kappa)|^2$ to P'_z, on use of (25). For $m > 0$ the last term leads to a special case of the Weber–Schafheitlin integral:

$$\int_0^{\infty} \mathrm{d}\rho\, \rho^{-1} J_m(\kappa\rho) J_m(\kappa'\rho) = (2m)^{-1} \left[\frac{\min(\kappa, \kappa')}{\max(\kappa, \kappa')}\right]^m \tag{31}$$

(see [8, section 13.42, equation (1)]). The middle terms can be evaluated in terms of the more general integral formula 11.4.33 of [11], resulting in exact cancellation of the term arising from (31). Thus we are left with

$$cP'_z = \frac{1}{4k} \int_0^k \mathrm{d}\kappa\, \kappa q |f(\kappa)|^2. \tag{32}$$

The results (26) and (32) are in accord with the inequality $U' \geqslant cP'_z$ proved in [2], since $q = \sqrt{k^2 - \kappa^2} \leqslant k$, the range of κ having been constrained to $(0, k)$ in (14) to avoid exponential growth of the wavefunction with z.

The same methods enable us to evaluate the invariant T'_{zz} arising from the conservation of momentum. We find

$$\begin{aligned} T'_{zz} &= \frac{1}{16\pi} \int \mathrm{d}^2 r (|E_\rho|^2 + |B_\rho|^2 + |E_\phi|^2 + |B_\phi|^2 - |E_z|^2 - |B_z|^2) \\ &= \frac{1}{4k^2} \int_0^k \mathrm{d}\kappa\, \kappa q^2 |f(\kappa)|^2. \end{aligned} \tag{33}$$

The two other invariants arising from the conservation of momentum (T'_{xz} and T'_{yz}) are zero for TM beams based on wavefunctions with $\mathrm{e}^{\mathrm{i}m\phi}$ azimuthal dependence (see section 3 of [2]).

Likewise two of the three invariants due to the conservation of angular momentum are zero. The non-zero invariant is, from [2] and (19) (note that B_z is zero for TM beams),

$$\begin{aligned} M'_{zz} &= \int \mathrm{d}^2 r\, [x\bar{\tau}_{yz} - y\bar{\tau}_{xz}] \\ &= -\frac{1}{8\pi} \int \mathrm{d}^2 r\ \mathrm{Re}[(xE_y - yE_x)E^*_z] \\ &= -\frac{1}{8\pi} \int \mathrm{d}^2 r\, \rho\, \mathrm{Re}(E_\phi E^*_z) \\ &= \frac{1}{8\pi} \frac{m}{k^2} \int \mathrm{d}^2 r\ \mathrm{Im}\{(\partial_z \psi) \cdot (\partial_z^2 + k^2)\psi^*\}. \end{aligned} \tag{34}$$

Now when ψ is given by (14) we have

$$\begin{aligned} \partial_z \psi &= \mathrm{i}\mathrm{e}^{\mathrm{i}m\phi} \int_0^k \mathrm{d}\kappa\, q f(\kappa) \mathrm{e}^{\mathrm{i}qz} J_m(\kappa\rho) \\ (\partial_z^2 + k^2)\psi &= \mathrm{e}^{\mathrm{i}m\phi} \int_0^k \mathrm{d}\kappa\, \kappa^2 f(\kappa) \mathrm{e}^{\mathrm{i}qz} J_m(\kappa\rho). \end{aligned} \tag{35}$$

Thus the use of (25) gives

$$M'_{zz} = \frac{m}{4k^2}\int_0^k \mathrm{d}\kappa\,\kappa q|f(\kappa)|^2. \qquad (36)$$

Finally we shall evaluate the angular momentum content per unit length of the TM beam based on ψ_m of (14). The angular momentum density is, when cycle-averaged,

$$\bar{j}_z = \rho\bar{p}_\phi = \frac{\rho}{8\pi c}\,\mathrm{Re}[(\mathbf{E}\times\mathbf{B}^*)_\phi] = \frac{m}{8\pi ck}\,\mathrm{Re}\{\psi(\partial_z^2+k^2)\psi^*\} \qquad (37)$$

so the angular momentum content per unit length of the beam is

$$J'_z = \frac{m}{4ck}\int_0^k \mathrm{d}\kappa\,\kappa|f(\kappa)|^2. \qquad (38)$$

Comparison of (26) with (38) and of (32) with (36) shows that

$$\omega J'_z = mU' \qquad \text{and} \qquad kM'_{zz} = mcP'_z. \qquad (39)$$

These equalities relate the angular momentum to the energy, and the angular momentum flux to the momentum. They are consistent with the picture of the beam consisting of photons of energy $\hbar\omega$, momentum $\hbar\omega/c$ and angular momentum $\hbar m$.

4. Invariants of TE + iTM beams

The TE + iTM beam complex magnetic field was given in (11). The complex electric field is $\mathbf{E}(\mathbf{r}) = \mathrm{i}\mathbf{B}(\mathbf{r})$; the field is *steady*, so equations (13) give the energy and momentum densities. For wavefunctions like (14) with $\mathrm{e}^{\mathrm{i}m\phi}$ azimuthal dependence, we have

$$\begin{aligned}\partial_x\psi &= (\cos\phi\,\partial_\rho - \mathrm{i}m\rho^{-1}\sin\phi)\psi\\ \partial_y\psi &= (\sin\phi\,\partial_\rho + \mathrm{i}m\rho^{-1}\cos\phi)\psi.\end{aligned} \qquad (40)$$

The radial, azimuthal and longitudinal components of $\mathbf{B}(\mathbf{r})$ are therefore

$$\begin{aligned}B_\rho &= \cos\phi B_x + \sin\phi B_y = k^{-1}\partial_\rho\partial_z\psi + \mathrm{i}m\rho^{-1}\psi\\ B_\phi &= -\sin\phi B_x + \cos\phi B_y = (k\rho)^{-1}\mathrm{i}m\partial_z\psi - \partial_\rho\psi\\ B_z &= k^{-1}\partial_z^2\psi + k\psi.\end{aligned} \qquad (41)$$

The energy density is $(8\pi)^{-1}[|B_\rho|^2 + |B_\phi|^2 + |B_z|^2]$; the z-component of the momentum density is $(\mathrm{i}/8\pi c)(B_\rho B_\phi^* - B_\phi B_\rho^*)$. The methods of the previous section give us, for the TE + iTM beam with ψ_m,

$$\begin{aligned}U' &= \tfrac{1}{2}\int_0^k \mathrm{d}\kappa\,\kappa|f(\kappa)|^2\\ cP'_z &= \frac{1}{2k}\int_0^k \mathrm{d}\kappa\,\kappa q|f(\kappa)|^2.\end{aligned} \qquad (42)$$

The inequality $U' \geqslant cP'_z$ [2] is satisfied, since $q \leqslant k$. Note that these values of energy and momentum content per unit length are twice those of the TM beam.

The angular momentum density is independent of time, since the momentum density is independent of time. It is given by

$$j_z = \rho p_\phi = \frac{\rho}{8\pi c}\mathrm{i}(\mathbf{B}\times\mathbf{B}^*)_\phi = \frac{\rho}{4\pi c}\,\mathrm{Im}(B_\rho B_z^*). \qquad (43)$$

We find (see appendix C)

$$\omega J'_z = \frac{1}{2k}\int_0^k \mathrm{d}\kappa\,\kappa(mk + \kappa^2/2q)|f(\kappa)|^2. \qquad (44)$$

To calculate the invariants consequent on the conservation of momentum we need (in general) to evaluate integrals over three of the cycle-averaged elements of the momentum flux density tensor (the negative of the Maxwell stress tensor), namely

$$\begin{aligned}\bar{\tau}_{xz} &= -\frac{1}{8\pi}\,\mathrm{Re}(E_x E_z^* + B_x B_z^*)\\ \bar{\tau}_{yz} &= -\frac{1}{8\pi}\,\mathrm{Re}(E_y E_z^* + B_y B_z^*)\\ \bar{\tau}_{zz} &= \frac{1}{16\pi}\{|E_x|^2 + |E_y|^2 - |E_z|^2 + |B_x|^2 + |B_y|^2 - |B_z|^2\}.\end{aligned} \qquad (45)$$

In the case of *steady beams*, with complex fields related by $\mathbf{E} = \pm\mathrm{i}\mathbf{B}$, the momentum flux density tensor is time-independent, and cycle-averaging is not necessary; the formulae (45) simplify to

$$\begin{aligned}&\tau_{xz} = -\frac{1}{4\pi}\,\mathrm{Re}(B_x B_z^*), \qquad \tau_{yz} = -\frac{1}{4\pi}\,\mathrm{Re}(B_y B_z^*)\\ &\tau_{zz} = \frac{1}{8\pi}\{|B_x|^2 + |B_y|^2 - |B_z|^2\} = \frac{1}{8\pi}\{|B_\rho|^2 + |B_\phi|^2 - |B_z|^2\}.\end{aligned} \qquad (46)$$

Since $B_x = \cos\phi B_\rho - \sin\phi B_\phi$ and $B_y = \sin\phi B_\rho + \cos\phi B_\phi$, we see from the TE + iTM field components given in (41) that wavefunctions with $\mathrm{e}^{\mathrm{i}m\phi}$ dependence will have τ_{xz} and τ_{yz} linear in $\sin\phi$ and $\cos\phi$, with no other azimuthal dependence, and so the integral invariants $T'_{xz} = \int \mathrm{d}^2r\,\bar{\tau}_{xz}$ and $T'_{yz} = \int \mathrm{d}^2r\,\bar{\tau}_{yz}$ will be zero. The remaining integral invariant arising out of the conservation of momentum is, for the TE + iTM beam with wavefunction (14),

$$T'_{zz} = \int \mathrm{d}^2r\,\tau_{zz} = \frac{1}{2k^2}\int_0^k \mathrm{d}\kappa\,\kappa q^2|f(\kappa)|^2. \qquad (47)$$

The invariants arising out of the conservation of angular momentum are [2]

$$\begin{aligned}M'_{zx} &= \int \mathrm{d}^2r\,[y\bar{\tau}_{zz} - z\bar{\tau}_{yz}]\\ M'_{zy} &= \int \mathrm{d}^2r\,[z\bar{\tau}_{xz} - x\bar{\tau}_{zz}]\\ M'_{zz} &= \int \mathrm{d}^2r\,[x\bar{\tau}_{yz} - y\bar{\tau}_{xz}].\end{aligned} \qquad (48)$$

For the TE + iTM beam under consideration, τ_{xz} and τ_{yz} are linear in $\sin\phi$ and $\cos\phi$, so M'_{zx} and M'_{zy} are zero. The integrand of the M'_{zz} invariant is

$$\begin{aligned}x\tau_{yz} - y\tau_{xz} &= \frac{\rho}{4\pi}\,\mathrm{Re}\{(\sin\phi B_x - \cos\phi B_y)B_z^*\}\\ &= -\frac{\rho}{4\pi}\,\mathrm{Re}(B_\phi B_z^*).\end{aligned} \qquad (49)$$

This gives us the remaining non-zero invariant,

$$M'_{zz} = \frac{m}{2k^2}\int_0^k \mathrm{d}\kappa\,\kappa q|f(\kappa)|^2. \qquad (50)$$

Note the proportionality between the momentum content per unit length and the angular momentum flux density integrated over a beam section:

$$mcP'_z = kM'_{zz}. \qquad (51)$$

5. Invariants of the 'CP' beam

The textbook circularly polarized 'beam' is not a finite beam but the transversely infinite plane wave

$$\mathbf{E}(\mathbf{r}) = E_0 e^{ikz}(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}), \qquad \mathbf{B}(\mathbf{r}) = \mp i\mathbf{E}(\mathbf{r}) \tag{52}$$

in which the electric and magnetic vectors rotate (at fixed z) in the xy plane as time progresses. Theorem 2.3 of [7] shows that finite beams cannot be everywhere exactly circularly polarized in a fixed plane; hence the quotation marks around 'CP'. Beams which are approximately circularly polarized (and come closer to pure circular polarization as the beam gets wider) were discussed in section 4 of [7]. Those with positive helicity have the vector potential given in (11), namely

$$\mathbf{A} = k^{-1}[-(\partial_z + ik), -i(\partial_z + ik), \partial_x + i\partial_y]\psi. \tag{53}$$

Where ψ is well represented by e^{ikz}, the fields resulting from (53) are as given by the upper sign in (52). For general ψ we have

$$(\partial_x + i\partial_y)\psi = e^{i\phi}(\partial_\rho + i\rho^{-1}\partial_\phi)\psi \tag{54}$$

and this becomes $e^{i\phi}(\partial_\rho - m\rho^{-1})\psi$ for wavefunctions with azimuthal dependence given by $e^{im\phi}$. For such wavefunctions the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ has components given by

$$\begin{aligned} kB_\rho &= ie^{i\phi}\{\partial_z(\partial_z + ik) + \rho^{-1}\partial_\rho + m\rho^{-1}(\partial_\rho - \rho^{-1}) - m^2\rho^{-2}\}\psi \\ kB_\phi &= -e^{i\phi}\{\partial_z(\partial_z + ik) + \partial_\rho^2 - m\rho^{-1}(\partial_\rho - \rho^{-1})\}\psi \\ kB_z &= -ie^{i\phi}(\partial_\rho - m\rho^{-1})(\partial_z + ik)\psi. \end{aligned} \tag{55}$$

Note that the azimuthal dependence is exclusively in a phase factor in these field components. The energy and momentum densities are consequently independent of ϕ. For steady beams with $\mathbf{E} = i\mathbf{B}$ these are

$$\begin{aligned} u &= \frac{1}{8\pi}\{|B_\rho|^2 + |B_\phi|^2 + |B_z|^2\} \\ cp_z &= \frac{i}{8\pi}(B_\rho B_\phi^* - B_\phi B_\rho^*). \end{aligned} \tag{56}$$

From (55) and (56) we find

$$U' = \int d^2r\, u = \tfrac{1}{2}\int_0^k d\kappa\, \kappa^{-1}(k+q)^2|f(\kappa)|^2 \tag{57}$$

$$cP_z' = c\int d^2r\, p_z = \frac{1}{2k}\int_0^k d\kappa\, \kappa^{-1}q(k+q)^2|f(\kappa)|^2. \tag{58}$$

Again the inequality $cP_z' \leqslant U'$ is satisfied, since $q = \sqrt{k^2-\kappa^2} \leqslant k$.

The angular momentum density is given by (43) and (55). The consequent angular momentum content per unit length of the beam is

$$\omega J_z' = \tfrac{1}{2}\int_0^k d\kappa\, \kappa^{-1}(k+q)^2\{m + 1 + \kappa^2/2kq\}|f(\kappa)|^2. \tag{59}$$

The invariants associated with the conservation of momentum are integrals over the momentum flux density tensor elements τ_{xz}, τ_{yz} and τ_{zz}, given in (46). Since

$$B_x = \cos\phi B_\rho - \sin\phi B_\phi, \qquad B_y = \sin\phi B_\rho + \cos\phi B_\phi \tag{60}$$

we see from (46) and (55) that the integrals over the azimuthal angle of τ_{xz} and τ_{yz} will be zero, as they were for the TE + iTM beam. This leaves one non-zero invariant associated with momentum conservation, namely

$$T_{zz}' = \int d^2r\, \tau_{zz} = \frac{1}{2k^2}\int_0^k d\kappa\, \kappa^{-1}q^2(k+q)^2|f(\kappa)|^2. \tag{61}$$

The invariants associated with the conservation of angular momentum are given in (48). Again only M_{zz}' of the three invariants is non-zero, because of the azimuthal dependence of τ_{xz} and τ_{yz}. We find, from (49) and (55),

$$\begin{aligned} M_{zz}' &= -\frac{1}{4\pi}\int d^2r\, \rho\, \mathrm{Re}(B_\phi B_z^*) \\ &= \frac{m+1}{2k^2}\int_0^k d\kappa\, \kappa^{-1}q(k+q)^2|f(\kappa)|^2. \end{aligned} \tag{62}$$

These results will be discussed and compared with those for other beams in the next section.

6. Summary and discussion

Barnett and Allen [1] wrote down a solution of Maxwell's equations in terms of a superposition of Bessel beams [12, 13]. Bessel beams are based on scalar solutions of the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$ of the form $e^{iqz}J_m(\kappa\rho)e^{im\phi}$, where $\kappa^2 + q^2 = k^2$. These solutions correspond to a delta-function amplitude $f(\kappa)$ in the wavefunction (14), leading to divergent integrals for all of the physical quantities considered here, such as the energy content per unit length of the beam. They are thus physically unrealizable, although finite-aperture truncations are realizable (see for example [14]). The wavefunctions of the Barnett and Allen type do result in convergent integrals for these physical quantities, provided the relevant integrals over $|f(\kappa)|^2$ are convergent.

We have calculated five invariants for three types of beams, based on wavefunctions expressed as superpositions of Bessel beams. The results are summarized in table 1.

The three types of beams we have considered share some remarkable properties:

(i) There are two more non-zero invariants than expected (only P_z', T_{zz}' and M_{zz}' are known to be independent of z for an arbitrary beam [2]).

(ii) The results for U', P_z' and T_{zz}' are all independent of m. For example, an infinity of beams, with $m = 0, \pm1, \pm2, \ldots$ in the wavefunction (14) share the same value for the energy content per unit length of the beam.

(iii) The TM (or TE) beam shares four of its five invariant values with those of the coherent superposition $(\mathrm{TE} + i\mathrm{TM})/\sqrt{2}$ (the factor $1/\sqrt{2}$ has been inserted to give the same normalization).

(iv) The TM and TE beams have energy and angular momentum contents in accord with the customary view of beams consisting of photons each carrying energy $\hbar\omega$ and angular momentum $\hbar m$ (when the wavefunction has azimuthal dependence $e^{im\phi}$). The other beams have a more complicated relationship between energy and angular momentum.

Table 1. Invariants of three types of generalized Bessel beams. The entries in the table give the multiplier M in the integral $\frac{1}{4}\int_0^k d\kappa\,\kappa|f(\kappa)|^2 M$; $\gamma = (k+q)^2/\kappa^2$.

	U'	cP'_z	$\omega J'_z$	T'_{zz}	kM'_{zz}
TM or TE	1	q/k	m	q^2/k^2	mq/k
(TE + iTM)/$\sqrt{2}$	1	q/k	$m+\kappa^2/2kq$	q^2/k^2	mq/k
'CP'/$\sqrt{2}$	γ	$\gamma q/k$	$\gamma[m+1+\kappa^2/2kq]$	$\gamma q^2/k^2$	$\gamma(m+1)q/k$

(v) The 'circularly polarized' beams have M'_{zz} proportional to $m+1$ rather than to m, for azimuthal dependence $e^{im\phi}$. This suggests an orbital component ($\hbar m$ in photon terms) and a spin component ($\hbar$) in the angular momentum of these beams. Such orbital-spin decomposition is discussed in the reprints in section 2 of [15]. Note however that the m-dependence of J'_z is more complicated.

Appendix A. U' for the TM beam

The energy content per unit length of the beam is, from equations (18), (19) and (21),

$$U' = \frac{1}{8k^2}\int_0^\infty d\rho\,\rho\int_0^k d\kappa\, f(\kappa) \times \int_0^k d\kappa'\, f^*(\kappa')e^{i(q-q')z}\{\text{integrand}\} \tag{A.1}$$

where the integrand is the sum of three terms, namely

$$\kappa\kappa'[\kappa\kappa' J_m(\kappa\rho)J_m(\kappa'\rho) + (k^2+qq')J_{m+1}(\kappa\rho)J_{m+1}(\kappa'\rho)] - m\rho^{-1}(k^2+qq')[\kappa J_{m+1}(\kappa\rho)J_m(\kappa'\rho) + \kappa' J_m(\kappa\rho)J_{m+1}(\kappa'\rho)] + 2m^2\rho^{-2}(k^2+qq')J_m(\kappa\rho)J_m(\kappa'\rho). \tag{A.2}$$

The first term can be integrated with the help of (25), and yields (26). The second and third terms are zero when $m = 0$. For $m > 0$ integral over ρ of the third term is

$$m(k^2+qq')\left[\frac{\min(\kappa,\kappa')}{\max(\kappa,\kappa')}\right]^m \tag{A.3}$$

on using (31). To evaluate the integral over ρ of the second term we use the Weber–Schafheitlin type integral 11.4.33 of [11]. The result is the negative of (A.3), i.e. the integrals of the second and third terms cancel exactly.

Appendix B. Singular integrals over products of Bessel functions

We consider the singular integral (25) as a limit as $a \to \infty$ of

$$W_m(a;\kappa,\kappa') \equiv \int_0^\infty d\rho\,\rho e^{-\rho^2/a^2}J_m(\kappa\rho)J_m(\kappa'\rho) = \tfrac{1}{2}a^2 e^{-(\kappa^2+\kappa'^2)a^2/4}I_m(\tfrac{1}{2}\kappa\kappa' a^2) \tag{B.1}$$

where I_m is the modified Bessel function of order m. The result (B.1) is known as Weber's second exponential integral, and is discussed in section 13.31 of [8]. The leading term in the asymptotic expansion of $I_m(z)$ is $e^z/\sqrt{2\pi z}$, so as a tends to infinity the right-hand side of (B.1) tends to the asymptotic value

$$W_a = \tfrac{1}{2}|a|(\pi\kappa\kappa')^{-\frac{1}{2}}\exp[-(\kappa-\kappa')^2a^2/4]. \tag{B.2}$$

This expression peaks at $\kappa = \kappa'$, with maximum value proportional to $|a|$ and width of the peak proportional to $|a|^{-1}$. The limit as $a \to \infty$ is thus a delta function, in accord with (25):

$$\int_0^\infty d\rho\,\rho J_m(\kappa\rho)J_m(\kappa'\rho) = \lim_{a\to\infty} W_m(a;\kappa,\kappa') = (\kappa\kappa')^{-\frac{1}{2}}\delta(\kappa-\kappa'). \tag{B.3}$$

We shall also need the limit as a tends to infinity of the integrals

$$V_m(a;\kappa,\kappa') = \int_0^\infty d\rho\,\rho^2 e^{-\rho^2/a^2}J_{m+1}(\kappa\rho)J_m(\kappa'\rho). \tag{B.4}$$

Differentiation of (B.1) with respect to κ, with use of

$$\partial_\kappa J_m(\kappa\rho) = \frac{m}{\kappa}J_m(\kappa\rho) - \rho J_{m+1}(\kappa\rho) \tag{B.5}$$

gives us

$$V_m(a;\kappa,\kappa') = \left(\frac{m}{\kappa} - \partial_\kappa\right)W_m(a;\kappa,\kappa') = \tfrac{1}{4}a^4 e^{-(\kappa^2+\kappa'^2)a^2/4}\{\kappa I_m(\kappa\kappa' a^2/2) - \kappa' I_{m+1}(\kappa\kappa' a^2/2)\}. \tag{B.6}$$

In performing first the differentiation and then the limiting process of letting a tend to infinity, we need to include the next term in the asymptotic expansion of I_m: we use

$$W_m(a;\kappa,\kappa') \sim W_a\left\{1 - \frac{4m^2-1}{4\kappa\kappa' a^2} + \cdots\right\} \tag{B.7}$$

and find the corresponding leading asymptotic parts of V_m:

$$V_m(a;\kappa,\kappa') \sim W_a\{\tfrac{1}{2}(\kappa-\kappa')a^2 + \tfrac{1}{4}(m+\tfrac{1}{2}) \times [\kappa + 3\kappa' - 2m(\kappa-\kappa')]/\kappa\kappa' + O(a^{-2})\}. \tag{B.8}$$

Appendix C. J'_z for the TE + iTM beam

The angular momentum density is, from (41) and (43),

$$j_z = \frac{\rho}{4\pi ck}\,\mathrm{Im}\{(k^{-1}\partial_\rho\partial_z\psi + im\rho^{-1}\psi)(\partial_z^2\psi^* + k^2\psi^*)\}. \tag{C.1}$$

Use of (23) and the first part of (35) gives

$$k^{-1}\partial_\rho\partial_z\psi + im\rho^{-1}\psi = ie^{im\phi}\int_0^k d\kappa\, f(\kappa)e^{iqz} \times \left\{m\rho^{-1}\left(1+\frac{q}{k}\right)J_m(\kappa\rho) - \frac{\kappa q}{k}J_{m+1}(\kappa\rho)\right\}. \tag{C.2}$$

Thus, on use of the second part of (35), the angular momentum density simplifies to

$$j_z = \frac{\rho}{4\pi ck} \mathrm{Re} \left\{ \int_0^k \mathrm{d}\kappa\, f(\kappa) \mathrm{e}^{\mathrm{i}qz} \left[m\rho^{-1} \left(1 + \frac{q}{k}\right) J_m(\kappa\rho) - \frac{\kappa q}{k} J_{m+1}(\kappa\rho) \right] \int_0^k \mathrm{d}\kappa' \,(\kappa')^2 f^*(\kappa') \mathrm{e}^{-\mathrm{i}q'z} J_m(\kappa'\rho) \right\}. \quad \text{(C.3)}$$

The angular momentum content per unit length of the beam is therefore

$$J_z' = \frac{1}{2ck} \mathrm{Re} \left\{ \int_0^\infty \mathrm{d}\rho\, \rho \int_0^k \mathrm{d}\kappa\, f(\kappa) \times \int_0^k \mathrm{d}\kappa' \,(\kappa')^2 f^*(\kappa') \mathrm{e}^{\mathrm{i}(q-q')z} [\text{integrand}] \right\} \quad \text{(C.4)}$$

where the integrand equals

$$\left[\left(1 + \frac{q}{k}\right) m J_m(\kappa\rho) - \frac{\kappa q}{k} \rho J_{m+1}(\kappa\rho) \right] J_m(\kappa'\rho). \quad \text{(C.5)}$$

Use of (25) evaluates the contribution of the $J_m(\kappa\rho)J_m(\kappa'\rho)$ term to J_z' as

$$\left(\frac{m}{2ck^2}\right) \int_0^k \mathrm{d}\kappa\, \kappa(k+q)|f(\kappa)|^2. \quad \text{(C.6)}$$

For the remaining part we need to make use of the limit as $a \to \infty$ of the integrals V_m of appendix B. The contribution to J_z' contains two types of delta-function terms:

$$\lim_{a\to\infty} \frac{|a|}{2\sqrt{\pi}} \exp[-(\kappa - \kappa')^2 a^2/4] = \delta(\kappa - \kappa') \quad \text{(C.7)}$$

which we saw in (B.3), and also

$$\lim_{a\to\infty} \frac{|a|^3}{4\sqrt{\pi}} (\kappa - \kappa')^2 \exp[-(\kappa - \kappa')^2 a^2/4] = \delta(\kappa - \kappa'). \quad \text{(C.8)}$$

(The function on the left of (C.8) peaks at $\kappa - \kappa' = \pm 2/|a|$, with maximum value proportional to $|a|$ and width of the peaks proportional to $|a|^{-1}$; it may be regarded as two delta functions of strength $1/2$, spaced infinitesimally to the left and right of $\kappa = \kappa'$.)

The contribution of the V_m integrals to J_z' is of the form

$$-\frac{1}{4ck^2} \int_0^k \mathrm{d}\kappa\, \kappa q f(\kappa) \mathrm{e}^{\mathrm{i}qz} \times \int_0^k \mathrm{d}\kappa'\, \kappa'^2 f^*(\kappa') \mathrm{e}^{-\mathrm{i}q'z} V_m(a;\kappa,\kappa') + \text{c.c.}$$

$$= -\frac{1}{4ck^2} \int_0^k \mathrm{d}\kappa \int_0^k \mathrm{d}\kappa'\, f(\kappa) f^*(\kappa') \kappa\kappa' \mathrm{e}^{\mathrm{i}(q-q')z} \times \left\{ q\kappa' V_m(a;\kappa,\kappa') + q'\kappa V_m(a;\kappa',\kappa) \right\}. \quad \text{(C.9)}$$

The asymptotic expansion as $a \to \infty$ gives us the two delta functions (C.7) and (C.8), the contribution to J_z' being

$$\frac{-m}{2ck^2} \int_0^k \mathrm{d}\kappa\, \kappa \left(mq - \frac{\kappa^2}{2q} \right) |f(\kappa)|^2. \quad \text{(C.10)}$$

The total angular momentum per unit length of the beam is thus

$$J_z' = \frac{m}{2ck^2} \int_0^k \mathrm{d}\kappa\, \kappa(mk + \kappa^2/2q)|f(\kappa)|^2. \quad \text{(C.11)}$$

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