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Forces on scatterers in particle beams

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Abstract
The force on a single scatterer can be calculated from the momentum transfer cross-section. A conglomerate of two or more scatterers will also have an effective momentum transfer cross-section, but this provides only the total force on the conglomerate. To calculate the forces on the individual scatterers we need to apply the recently introduced momentum flux density tensor (the quantum analogue of the Maxwell stress tensor in electrodynamics). We show that, for a single scatterer, the use of the momentum flux density tensor in the near field reproduces the force calculated from the far-field scattering amplitude.

1. Introduction

Suppose the interaction between a scatterer and the particles in a beam is central, i.e. given by a potential $V(r)$. Schrödinger's equation for the particles (of mass m and energy $E = \hbar^2 k^2/2m$) reads

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi = E\psi. \tag{1}$$

The solution outside the range of $V(r)$ appropriate to an incoming plane wave beam $\psi_0 = \mathrm{e}^{\mathrm{i}kz}$ (and to a scatterer held at the origin) is the well-known partial wave series [1]

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{i}^{\ell}\,\mathrm{e}^{\mathrm{i}\delta_\ell}[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)]P_\ell(\cos\theta) \tag{2}$$

where δ_ℓ are the phase shifts, j_ℓ and n_ℓ are spherical Bessel functions and P_ℓ are Legendre polynomials. The total wavefunction ψ is the sum of the incident wave

$$\mathrm{e}^{\mathrm{i}kz} = \sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{i}^{\ell} j_\ell(kr)P_\ell(\cos\theta) \tag{3}$$

and the scattered wave ψ_s, which has the far-field form

$$\psi_s(r,\theta) \to r^{-1} f(\theta)\,\mathrm{e}^{\mathrm{i}kr}. \tag{4}$$

The scattering amplitude $f(\theta)$ is given by

$$f(\theta) = k^{-1} \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell P_\ell(\cos\theta). \tag{5}$$

The force on the scatterer is the rate of transfer of momentum, by Newton's law. An incoming beam particle has momentum $\hbar k_z = \hbar k$; scattering through angle θ changes this to $\hbar k'_z = \hbar k \cos\theta$, so the momentum transfer to the scatterer is $\hbar k(1-\cos\theta)$ in the forward direction. The force on the scatterer is the rate of change of momentum within the beam,

$$\text{force} = n\frac{\hbar k}{m}\sigma_p \hbar k = 2nE\sigma_p \tag{6}$$

where n is the particle density in the beam, $\hbar k/m$ is their speed, $\hbar k$ is the momentum per particle and σ_p is the momentum transfer cross-section

$$\sigma_p = 2\pi \int_0^\pi d\theta \sin\theta (1-\cos\theta)|f(\theta)|^2. \tag{7}$$

With the use of (we write C for $\cos\theta$)

$$\begin{aligned} \int_{-1}^{1} dC\, P_\ell(C) P_{\ell'}(C) &= \frac{2}{2\ell+1}\delta_{\ell',\ell} \\ \int_{-1}^{1} dC\, C\, P_\ell(C) P_{\ell'}(C) &= \frac{2(\ell+1)}{(2\ell+1)(2\ell+3)}\delta_{\ell',\ell+1} + \frac{2\ell}{(2\ell-1)(2\ell+1)}\delta_{\ell',\ell-1} \end{aligned} \tag{8}$$

one finds [2]

$$\sigma_p = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (\ell+1) \sin^2(\delta_\ell - \delta_{\ell+1}). \tag{9}$$

When we consider extensions of this standard scattering theory to groups of two or more scatterers, it is immediately apparent that although the total force on the group may in principle be calculated from a generalized momentum transfer cross-section, the forces on individual scatterers cannot. This is because the new scattering amplitude $f(\theta, \phi)$ is found from the total far-field scattered part of the wavefunction, which is a coherent superposition of the waves originating in the scattering conglomerate, multiply scattered waves being included. Thus, the total wavefunction is truly 'entangled', and the force on an individual scatterer cannot be disentangled from the far-field scattering amplitude.

On the other hand, one may draw a closed surface (not necessarily a sphere) around an individual scatterer, and calculate the rate of momentum transfer through this surface by means of the momentum flux density tensor. In general, such a computation may have to be done numerically, but by this means it is possible in principle to calculate the force acting on each of the scatterers. To demonstrate the validity of such an approach, we shall verify in this note that the momentum flux density tensor method reproduces the force on an isolated scatterer, equations (6) and (9).

2. Force from the momentum flux density tensor

The momentum flux density tensor was introduced [3] in the context of momentum conservation in atomic beams. It is defined in analogy with the Maxwell stress tensor of electrodynamics [4, 5]. Let a particle field be characterized by the wavefunction $\Psi(\mathbf{r}, t)$, and define a real momentum density $\mathbf{p}(\mathbf{r}, t)$ for this state by [3]

$$\mathbf{p}(\mathbf{r}, t) = \tfrac{1}{2}\{\Psi^* \hat{\mathbf{p}}\Psi + \Psi(\hat{\mathbf{p}}\Psi)^*\} \tag{10}$$

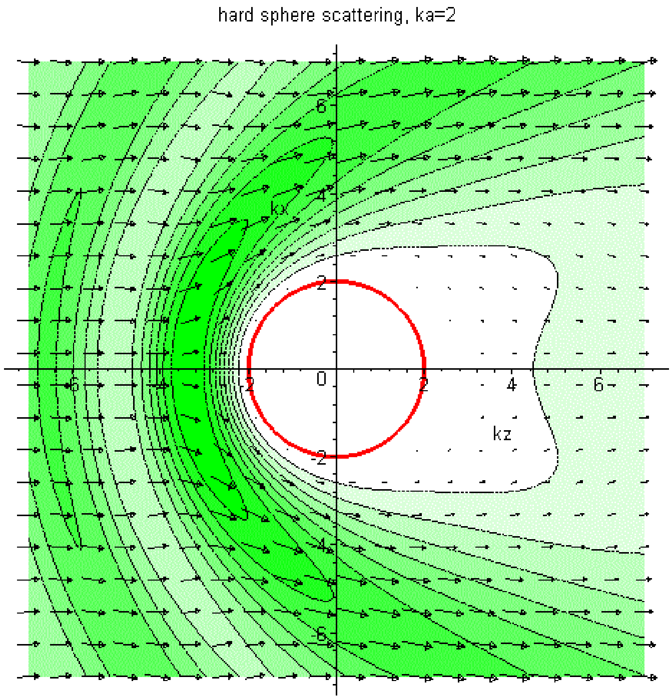

Figure 1. Scattering of a wide beam, incident from the left, by a hard sphere; $ka = 2$, so the wavelength in the undisturbed beam is π times the sphere radius. The contours and shading show the probability density $|\Psi|^2$; the arrows represent the momentum density or the probability density current (both of which are proportional to $\mathrm{Im}(\Psi^*\nabla\Psi)$). In this paper we show that the force on the scatterer may be calculated as an integral over a closed surface drawn anywhere outside the range of the scatterer potential (in this case, anywhere outside $r = a$).

where $\hat{\mathbf{p}}$ is the momentum operator $-\mathrm{i}\hbar\nabla$; thus

$$\mathbf{p}(\mathbf{r}, t) = \frac{\mathrm{i}\hbar}{2}\{\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\} = \hbar\,\mathrm{Im}(\Psi^*\nabla\Psi) = m\mathbf{J}(\mathbf{r}, t) \tag{11}$$

where $\mathbf{J}$ is the probability flux density. A direct consequence of Schrödinger's equation is the continuity equation [1]

$$\frac{\partial|\Psi|^2}{\partial t} + \nabla\cdot\mathbf{J} = 0. \tag{12}$$

Figure 1 shows the probability density $|\Psi|^2$ and the momentum density $\mathbf{p} = \hbar\,\mathrm{Im}(\Psi^*\nabla\Psi)$ in the neighborhood of an impenetrable sphere scattering a wide beam (i.e., one represented by the plane wave $\mathrm{e}^{\mathrm{i}kz}$ far from the scatterer).

In [3] it is shown that the conservation of momentum is expressed in three equations of the same form, namely

$$\frac{\partial p_i}{\partial t} + \sum_j \partial_j \tau_{ij} = f_i, \qquad (i, j = x, y, z) \tag{13}$$

where τ_{ij} are elements of a momentum flux density tensor,

$$\tau_{ij} = \frac{\hbar^2}{2m}\left\{(\partial_i\Psi^*)(\partial_j\Psi) + (\partial_i\Psi)(\partial_j\Psi^*) - \frac{1}{2}\partial_i\partial_j(\Psi^*\Psi)\right\}$$
$$= \frac{\hbar^2}{2m}\,\mathrm{Re}\{(\partial_i\Psi^*)(\partial_j\Psi) - \Psi^*\partial_i\partial_j\Psi\}. \tag{14}$$

This tensor is real and symmetric. It is different from the 'stress tensor for the probability fluid', introduced by Madelung in 1926 [6, 7],

$$T_{ij} = \frac{\hbar^2}{4m}[\partial_i\partial_j(\Psi^*\Psi) - (\Psi^*\Psi)^{-1}\partial_i(\Psi^*\Psi)\partial_j(\Psi^*\Psi)]. \tag{15}$$

In the conservation law (13), f_i stands for a component of the force density. We shall calculate the z-component of the force on a scatterer in a beam propagating in the z direction. If the beam does not fluctuate we have $\partial p_i/\partial t = 0$, i.e. the momentum density is a constant vector at a given point in space. The z-component of the total force exerted on the particles in the beam by the scatterer is thus

$$F_z = \int \mathrm{d}^3r\, f_z = \sum_j \int \mathrm{d}^3r\, \partial_j\tau_{jz} = \sum_j \int \mathrm{d}S_j\,\tau_{jz} \tag{16}$$

where the integration is over some volume including the scatterer, and then over the surface bounding this volume (these integrals are equal by the tensor generalization of the Gauss divergence theorem).

3. Force on a spherical scatterer

We now specialize to a spherical surface, at radius r from the centre of the scatterer. The surface area element is

$$\mathrm{d}\mathbf{S} = \hat{\mathbf{r}}r^2\,\mathrm{d}\Omega = r[x, y, z]\,\mathrm{d}\Omega \tag{17}$$

where $\mathrm{d}\Omega$ is an element of solid angle, and $\hat{\mathbf{r}} = \mathbf{r}/r$. Thus

$$F_z = r\int \mathrm{d}\Omega\{x\tau_{xz} + y\tau_{yz} + z\tau_{zz}\}. \tag{18}$$

Let $\psi(r,\theta)$ be the total wavefunction, as given in (2), of the beam plus the scattered wave. In the steady state being considered, $\Psi(\mathbf{r},t) = \psi(r,\theta)\,\mathrm{e}^{-\mathrm{i}\omega t}$ and the only time dependence is in the phase factor. Thus, for example,

$$\tau_{xz} = \frac{\hbar^2}{2m}\left\{(\partial_x\psi^*)(\partial_z\psi) + (\partial_z\psi^*)(\partial_x\psi) - \frac{1}{2}\partial_x\partial_z(\psi^*\psi)\right\}$$
$$= \frac{\hbar^2}{2m}\cos\phi\left\{(\partial_\rho\psi^*)(\partial_z\psi) + (\partial_z\psi^*)(\partial_\rho\psi) - \frac{1}{2}\partial_\rho\partial_z(\psi^*\psi)\right\} \tag{19}$$

where $\rho^2 = x^2 + y^2$, and we have used the fact that ψ is independent of the azimuthal angle ϕ, and the first of the pair

$$\partial_x = \cos\phi\,\partial_\rho - \frac{\sin\phi}{\rho}\partial_\phi, \qquad \partial_y = \sin\phi\,\partial_\rho + \frac{\cos\phi}{\rho}\partial_\phi. \tag{20}$$

Since $x = \rho\cos\phi$, $y = \rho\sin\phi$, the cylindrical-polar version of $\frac{2m}{\hbar^2}[x\tau_{xz} + y\tau_{yz} + z\tau_{zz}]$ is

$$\rho\left[(\partial_\rho\psi^*)(\partial_z\psi) + (\partial_z\psi^*)(\partial_\rho\psi) - \tfrac{1}{2}\partial_\rho\partial_z(\psi^*\psi)\right] + z\left[2(\partial_z\psi^*)(\partial_z\psi) - \tfrac{1}{2}\partial_z^2(\psi^*\psi)\right]. \tag{21}$$

Because of the assumed spherical symmetry of an individual scatterer, it is convenient to further transform to spherical polars, via

$$\partial_z = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta, \qquad \partial_\rho = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta. \tag{22}$$

The expression for $Z = \frac{2m}{\hbar^2}[x\tau_{xz} + y\tau_{yz} + z\tau_{zz}]$ then becomes

$$\begin{aligned} Z &= r\cos\theta\left\{|\partial_r\psi|^2 - \mathrm{Re}\left(\psi^*\partial_r^2\psi\right)\right\} + \sin\theta\,\mathrm{Re}\left\{\psi^*(\partial_r\partial_\theta\psi) - (\partial_r\psi^*)(\partial_\theta\psi) - \frac{1}{r}\psi^*(\partial_\theta\psi)\right\} \\ &= rC\left\{|\partial_r\psi|^2 - \mathrm{Re}\left(\psi^*\partial_r^2\psi\right)\right\} \\ &\quad - (1-C^2)\,\mathrm{Re}\left\{\psi^*(\partial_r\partial_C\psi) - (\partial_r\psi^*)(\partial_C\psi) - \frac{1}{r}\psi^*(\partial_C\psi)\right\} \end{aligned} \tag{23}$$

where $C = \cos\theta$ as in (8). (These expressions remain valid when ψ does depend on ϕ.) The total force is thus

$$F_z = r\frac{\hbar^2}{2m}\int \mathrm{d}\Omega\, Z = r\frac{\hbar^2}{2m}2\pi\int_0^\pi \mathrm{d}\theta\,\sin\theta\, Z = r\frac{\hbar^2}{2m}2\pi\int_{-1}^1 \mathrm{d}C\, Z. \tag{24}$$

Note that the force in (24) is the negative of the force on the scatterer, since it gives the total force on the beam.

In the absence of the scatterer $\psi \to \mathrm{e}^{\mathrm{i}kz} = \mathrm{e}^{\mathrm{i}krC}$, and $Z \to 2k^2rC$; the angular integral in (24) is thus zero, as it must be. However, one expects a zero contribution to the force, everywhere: when $\psi \to \mathrm{e}^{\mathrm{i}kz}$, $\tau_{xz}, \tau_{yz} \to 0$, $\tau_{zz} \to \frac{\hbar^2k^2}{m}$, and $f_z = \partial_x\tau_{xz} + \partial_y\tau_{yz} + \partial_z\tau_{zz} \to 0$, so the force density is zero. This apparent discrepancy (zero force density, but nonzero integrand in (24)) may be reconciled by noting that in the application of Gauss' divergence theorem one may add or subtract a vector of zero divergence:

$$\int \mathrm{d}\mathbf{S}\cdot(\mathbf{A}+\mathbf{B}) = \int \mathrm{d}V\,\nabla\cdot\mathbf{A} \qquad \text{when} \qquad \nabla\cdot\mathbf{B} = 0. \tag{25}$$

In the free beam case the only non-zero part of the tensor τ_{ij} is $\tau_{zz}^{(0)} = \hbar^2k^2/m$, a constant. The divergence free vector with elements $\tau_{iz}^{(0)}$ may be subtracted from τ_{iz} in Z, and so $2k^2rC = 2k^2z$ may be subtracted from (23), and then the integrand in the force expression (24) will be identically zero in the absence of the scatterer.

We now have two tasks: to verify (i) that (24) gives a total force which is independent of r, and (ii) that this force is in agreement with the known result contained in (6) and (9). (In both cases it is understood that the spherical surface we are integrating over is to be outside of the potential $V(r)$.) These tasks are carried out in the appendix, which verifies that the momentum flux density tensor route to the force reproduces the momentum transfer cross-section expression.

4. Discussion

We have applied the recently introduced momentum flux density tensor to the calculation of the force due to scattering. For a single spherical scatterer the net force was expressed as a surface integral involving spatial derivatives of the wavefunction. The integration can be over any closed surface surrounding the scatterer, outside the range of the interaction. For a spherical surface the force was shown to reduce to the known expression involving the momentum transfer cross-section, at any radius of the integration surface.

The method is intended for application to groups of scatterers, for which the momentum transfer cross-section method will give the net force on the group, but not the forces on the individual particles. Provided that the beam–scatterer interaction potential has finite range, and that the scatterers are separated by more than this range, it is possible to apply the momentum flux density tensor method to calculate the forces on each of the scatterers. A surface can be drawn around each scatterer, outside the range of its potential energy and of the potential energies of the other scatterers, and the sum $\sum_j \int \mathrm{d}S_j \tau_{ij}$ will give the ith component of the force exerted by that scatterer on the particles in the beam.

It is hoped that this formulation will prove useful in the study of the interaction of atom beams with scatterers [8]. It may even be possible to realize the atom-beam analogue of 'optical binding' [9, 10], in which dielectric particles in optical beams experience mutual binding forces. In fact, part of the motivation behind this work is to develop the momentum flux density tensor method (the quantum analogue of the Maxwell stress tensor) in order to be able to apply the electromagnetic analogue to the optical binding situation, especially in counter-propagating coherent beams [11].

Appendix. Evaluation of the force integral

We wish to evaluate the angular integral in (24), and then to simplify the resulting sum over partial waves. The wavefunction $\psi(r,\theta)$, given in (2), can be written as

$$\begin{aligned} \psi(r,\theta) &= \sum_{\ell=0}^{\infty} f_\ell(kr) P_\ell(\cos\theta), \\ f_\ell(kr) &= (2\ell+1)\mathrm{i}^\ell\, \mathrm{e}^{\mathrm{i}\delta_\ell}\left[\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell n_\ell(kr)\right]. \end{aligned} \tag{A.1}$$

We note that $\partial_C \psi$ has the angular parts

$$\frac{\mathrm{d}P_\ell}{\mathrm{d}C} = \frac{\ell+1}{1-C^2}(CP_\ell - P_{\ell+1}). \tag{A.2}$$

Thus, from (8) and (23), the integration over θ or $C = \cos\theta$ reduces the double sum over partial waves ℓ and ℓ' to a single sum, containing products of the ℓ and $\ell+1$ terms. We rewrite the second equality in (8) as

$$\int_{-1}^{1} \mathrm{d}C\, C P_\ell P_{\ell'} = \alpha_{\ell+1}\delta_{\ell',\ell+1} + \alpha_\ell \delta_{\ell',\ell-1}, \qquad \alpha_\ell = \frac{2\ell}{(2\ell-1)(2\ell+1)}. \tag{A.3}$$

We also write $\dot f_\ell(\rho)$ for $\mathrm{d}f_\ell(\rho)/\mathrm{d}\rho$, etc. Then $\partial_r\psi = k\Sigma \dot f_\ell P_\ell$ and

$$\begin{aligned} \int_{-1}^{1} \mathrm{d}C\, C|\partial_r\psi|^2 &= k^2 \sum_{\ell=0}^{\infty} \dot f_\ell^*(\alpha_{\ell+1}\dot f_{\ell+1} + \alpha_\ell \dot f_{\ell-1}) \\ &= k^2 \sum_{\ell=0}^{\infty} \alpha_{\ell+1}(\dot f_\ell^* \dot f_{\ell+1} + \dot f_{\ell+1}^* \dot f_\ell) \end{aligned} \tag{A.4}$$

since $\alpha_0 = 0$. Likewise

$$\begin{aligned} \int_{-1}^{1} \mathrm{d}C\, C\psi^* \partial_r^2 \psi &= k^2 \sum_{\ell=0}^{\infty} f_\ell^*(\alpha_{\ell+1}\ddot f_{\ell+1} + \alpha_\ell \ddot f_{\ell-1}) \\ &= k^2 \sum_{\ell=0}^{\infty} \alpha_{\ell+1}(f_\ell^* \ddot f_{\ell+1} + f_{\ell+1}^* \ddot f_\ell). \end{aligned} \tag{A.5}$$

We can write $f_\ell(kr)$ as a complex constant times a real function,

$$f_\ell = a_\ell g_\ell, \qquad a_\ell = (2\ell+1)i^\ell\, \mathrm{e}^{\mathrm{i}\delta_\ell}, \qquad g_\ell = \cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell n_\ell(kr). \tag{A.6}$$

The complex constants a_ℓ will occur in our force expression (24) in the combination $a_\ell^* a_{\ell+1}$ and its complex conjugate, both of which have the real part

$$\mathrm{Re}(a_\ell^* a_{\ell+1}) = \mathrm{Re}(a_\ell a_{\ell+1}^*) = (2\ell+1)(2\ell+3)\sin(\delta_\ell - \delta_{\ell+1}). \tag{A.7}$$

Also $\alpha_{\ell+1}(2\ell+1)(2\ell+3) = 2(\ell+1)$. Hence

$$\int_{-1}^{1} \mathrm{d}C C \left\{(\partial_r\psi)^2 - \mathrm{Re}\big(\psi^*\partial_r^2\psi\big)\right\}$$
$$= 2k^2 \sum_{\ell=0}^{\infty} (\ell+1)\sin(\delta_\ell - \delta_{\ell+1})\{2\dot g_\ell \dot g_{\ell+1} - g_\ell \ddot g_{\ell+1} - \ddot g_\ell g_{\ell+1}\}. \tag{A.8}$$

In the remaining part of (24) we have, using (A.2),

$$k\,\mathrm{Re}\int_{-1}^{1}\mathrm{d}C \sum_{\ell=0}^{\infty}\sum_{\ell'=0}^{\infty}(\ell+1)(CP_\ell - P_{\ell+1})P_{\ell'}[f_{\ell'}^*\dot f_\ell - \dot f_{\ell'}^* f_\ell - (kr)^{-1} f_{\ell'}^* f_\ell]$$
$$= k\,\mathrm{Re}\sum_{\ell=0}^{\infty}(\ell+1)\left\{\left(\alpha_{\ell+1} - \frac{2}{2\ell+3}\right)[f_{\ell+1}^*\dot f_\ell - \dot f_{\ell+1}^* f_\ell - (kr)^{-1} f_{\ell+1}^* f_\ell]\right.$$
$$\left. + \alpha_\ell [f_{\ell-1}^*\dot f_\ell - \dot f_{\ell-1}^* f_\ell - (kr)^{-1} f_{\ell-1}^* f_\ell]\right\}. \tag{A.9}$$

Now

$$(\ell+1)\left[\alpha_{\ell+1} - \frac{2}{2\ell+3}\right] = -\ell\alpha_{\ell+1}. \tag{A.10}$$

So, on changing the summation index in the α_ℓ term (as we did in (A.4) and (A.5)), (A.9) reduces to

$$k\,\mathrm{Re}\sum_{\ell=0}^{\infty}\alpha_{\ell+1}\{(\ell+2)[f_\ell^*\dot f_{\ell+1} - \dot f_\ell^* f_{\ell+1} - (kr)^{-1} f_\ell^* f_{\ell+1}]$$
$$- \ell[f_{\ell+1}^*\dot f_\ell - \dot f_{\ell+1}^* f_\ell - (kr)^{-1} f_{\ell+1}^* f_\ell]\}$$
$$= 2k\sum_{\ell=0}^{\infty}(\ell+1)\sin(\delta_\ell - \delta_{\ell+1})\{(\ell+2)[g_\ell \dot g_{\ell+1} - \dot g_\ell g_{\ell+1} - (kr)^{-1} g_\ell g_{\ell+1}]$$
$$- \ell[g_{\ell+1}\dot g_\ell - \dot g_{\ell+1} g_\ell - (kr)^{-1} g_{\ell+1} g_\ell]\}$$
$$= 2k\sum_{\ell=0}^{\infty}(\ell+1)\sin(\delta_\ell - \delta_{\ell+1})\{(2\ell+2)(g_\ell \dot g_{\ell+1} - \dot g_\ell g_{\ell+1}) - 2(kr)^{-1} g_\ell g_{\ell+1}\}. \tag{A.11}$$

Thus, the force integral $\int_{-1}^{1}\mathrm{d}C(23)$ is equal to

$$2k\sum_{\ell=0}^{\infty}(\ell+1)\sin(\delta_\ell - \delta_{\ell+1})\{kr[2\dot g_\ell \dot g_{\ell+1} - g_\ell \ddot g_{\ell+1} - \ddot g_\ell g_{\ell+1}]$$
$$- (2\ell+2)(g_\ell \dot g_{\ell+1} - \dot g_\ell g_{\ell+1}) + 2(kr)^{-1} g_\ell g_{\ell+1}\}. \tag{A.12}$$

The spherical Bessel functions $j_\ell(\rho)$ and $n_\ell(\rho)$ have the derivatives

$$\begin{aligned}
\dot b_\ell &= \frac{\ell}{\rho} b_\ell - b_{\ell+1}, & \ddot b_\ell &= \frac{2}{\rho} b_{\ell+1} - \left[1 - \frac{\ell(\ell-1)}{\rho^2}\right] b_\ell \\
\dot b_{\ell+1} &= b_\ell - \frac{\ell+2}{\rho} b_{\ell+1}, & \ddot b_{\ell+1} &= \frac{-2}{\rho} b_\ell - \left[1 - \frac{(\ell+2)(\ell+3)}{\rho^2}\right] b_{\ell+1}.
\end{aligned} \tag{A.13}$$

We also need the identity

$$j_{\ell+1}(\rho)n_\ell(\rho) - j_\ell(\rho)n_{\ell+1}(\rho) = \rho^{-2}. \tag{A.14}$$

Substitution into (A.12) reduces the contents of the braces to

$$2(kr)^{-1}(\sin\delta_{\ell+1}\cos\delta_\ell - \cos\delta_{\ell+1}\sin\delta_\ell) = -2(kr)^{-1}\sin(\delta_\ell - \delta_{\ell+1}) \tag{A.15}$$

Hence the force exerted on the beam by the scatterer is (for unit density of particles in the incident beam, i.e. with $|\psi|^2$ normalized to unity)

$$F_z = -\frac{\hbar^2 k^2}{2m} 2\sigma_p = -2E\sigma_p \tag{A.16}$$

where σ_p is the momentum transfer cross-section defined in (7) and (9). The above expression is in accord with the force on the scatterer given in (6).

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