Institute of Physics Publishing | European Journal of Physics
Eur. J. Phys. **25** (2004) 737–744 | PII: S0143-0807(04)82064-6

Electrostatics of hyperbolic conductors

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Received 14 June 2004
Published 25 August 2004
Online at stacks.iop.org/EJP/25/737
doi:10.1088/0143-0807/25/6/005

Abstract

Analytic expressions are found for the electric field and potential around a pair of hyperbolic conductors with a potential difference between them. The results also apply to the field and potential between a hyperbolic conductor and a conducting plane, and to the two-dimensional flow of an ideal fluid between hyperbolic barriers or between a flat surface and a hyperbolic barrier. The field strength at a conductor is found to be proportional to the cube root of the local curvature. (The planar case must be obtained as a limit.) The methods and results are simple enough to be used in teaching electrostatics and hydrodynamics, in particular to supply an explicit counter-example to the popular misconception that the field strength at a conductor is proportional to the local value of the curvature.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Richard Packard has proposed an experiment in which electrostrictively frozen helium would serve as an electrostatically variable barrier between two baths of liquid helium [1]. This paper presents a simple solvable two-dimensional model for the electrostatics and fluid mechanics of such a barrier. It may be useful in the analysis of the proposed experiment, and also as a worked example in teaching electrostatics and fluid mechanics, since all the desired characteristics (field lines, equipotentials, field strength, curvature of equipotential surface) may be obtained analytically by elementary mathematics.

It is well known that in electrostatics, in magnetostatics and in the flow of fluids with negligible compressibility and viscosity [2–5], the respective electric, magnetic and velocity potentials satisfy the Laplace equation. It is also well known (and easily verified) that in the two-dimensional case, any differentiable function w of the complex variable $z = x + \mathrm{i}y$ (or of $z^* = x - \mathrm{i}y$) will satisfy

$$\left(\partial_x^2 + \partial_y^2\right) w(x + \mathrm{i}y) = 0. \qquad (1)$$

The real and imaginary parts $u(x, y)$ and $v(x, y)$ of $w(z) = u + \mathrm{i}v$ also satisfy Laplace's equation, since by differentiating w with respect to x and y respectively we get

$$\partial_x w = w' = \partial_x u + \mathrm{i}\partial_x v \qquad \partial_y w = \mathrm{i}w' = \partial_y u + \mathrm{i}\partial_y v \tag{2}$$

and hence u and v satisfy the Cauchy–Riemann equations

$$\partial_x u = \partial_y v \qquad \partial_x v = -\partial_y u \tag{3}$$

from which the fact $\left(\partial_x^2 + \partial_y^2\right)(u, v) = 0$ follows by further differentiation.

Also, the curves along which u is constant are orthogonal to the curves along which v is constant: constant u implies $\partial_x u\,\mathrm{d}x + \partial_y u\,\mathrm{d}y = 0$, so the slope is $\mathrm{d}y/\mathrm{d}x = -\partial_x u/\partial_y u$. Likewise the slope of a $v =$ constant curve is $-\partial_x v/\partial_y v$, and the product of these two slopes is –1 by the Cauchy–Riemann equations, proving that the curves cross at right angles.

If the dimensionless function $v(x, y)$ is proportional to the potential, the dimensionless function $u(x, y)$ will give the electric field lines (or the magnetic field lines, or the stream lines of fluid flow). For, the electric lines of force are given by $\mathrm{d}x/E_x = \mathrm{d}y/E_y$, and $\mathbf{E}$ is proportional to the gradient of v, so $\partial_y v\,\mathrm{d}x = \partial_x v\,\mathrm{d}y$ on a line of force, or (by the Cauchy–Riemann equations) $\partial_x u\,\mathrm{d}x + \partial_y u\,\mathrm{d}y = 0$, which defines the curves of constant u.

A complex function $w = u + \mathrm{i}v$ representing electric potential and electric field between two hyperbolic conductors (or the velocity potential and stream function for flow between two hyperbolic barriers) will be given in the next section.

2. Hyperbolic conductors and barriers

Hyperbolae with asymptotes $y = \pm\frac{b}{a}x$ have the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{4}$$

The hyperbolae cross the x-axis at $x = \pm a$. The radius of curvature at these 'ends' is $\kappa^{-1} = b^2/a$. The foci of the hyperbolae are at $x = \pm\sqrt{a^2 + b^2}$. Sharp hyperbolae have $a > b$, blunt hyperbolae have $b > a$; the 'rectangular hyperbola' has $a = b$ (see for example [6]).

For our purposes, an important aspect is the parametric expression (see for example section 4.16 of [2]) for the right-hand branch of the hyperbola:

$$x = a\cosh\theta \qquad y = b\sinh\theta. \tag{5}$$

This is because (i) the functions $u(x, y)$ and $v(x, y)$ defined implicitly by

$$x + \mathrm{i}y = a\cosh(u + \mathrm{i}v) + \mathrm{i}b\sinh(u + \mathrm{i}v) \tag{6}$$

both satisfy the Laplace equation ($z = a\cosh w + \mathrm{i}b\sinh w$ defines an analytic function $w(z)$, with real and imaginary parts u and v). Further (ii) with v chosen as the potential function, $v = 0$ gives us $x = a\cosh u$, $y = b\sinh u$, i.e. the right-hand (positive x) branch of the hyperbola, as u varies from $-\infty$ to $+\infty$.

For general u and v, we obtain x and y by equating real and imaginary parts in (6):

$$x = C(ac - bs) \qquad y = S(as + bc) \tag{7}$$

where $C = \cosh u$, $S = \sinh u$, $c = \cos v$, $s = \sin v$. We can write (7) as $x = a'C$, $y = b'S$. If there is a value of v such that

$$a' = ac - bs \to -a \qquad b' = as + bc \to b \tag{8}$$

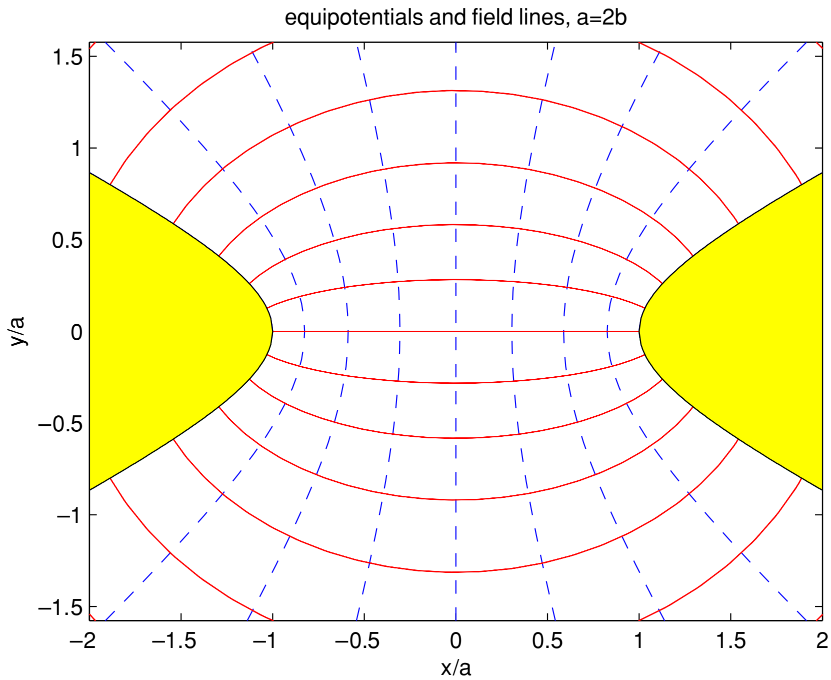

Figure 1. Two hyperbolic conductors, both with $a = 2b$. The equipotentials (dashed curves) are also hyperbolae; they are drawn for potential increments of $\Delta V/8$, where ΔV is the potential difference between the two conductors. The field lines (full curves) are ellipses, with semiaxes $R\cosh u$ and $R\sinh u$, where $R^2 = a^2 + b^2$. The field lines are drawn at equal intervals of u.

this value (v_0) would correspond to the potential at the left-hand hyperbolic conductor which is parametrized by $x = -a\cosh u$, $y = b\sinh u$. Equations (8) are solved by

$$c_0 = \cos v_0 = \frac{b^2 - a^2}{b^2 + a^2} \qquad \text{or} \quad v_0 = 2atn(a/b). \tag{9}$$

Also $x = 0$ when $\tan v = a/b$, i.e. when v is half of the above value, so the plane $x = 0$ is at a potential midway between that of the two hyperbolic conductors. If the physical complex potential is $U + \mathrm{i}V = U_0(u + \mathrm{i}v)$, the actual potential difference between the two hyperbolic conductors is

$$\Delta V = U_0 v_0 = 2U_0\, atn(a/b). \tag{10}$$

Figure 1 shows the equipotentials and field lines for a pair of hyperbolic conductors with $a = 2b$.

The second conductor does not have to be the hyperbola parametrized by $x = -a\cosh u$, $y = b\sinh u$. It could be the plane $x = 0$ at potential $\frac{1}{2}U_0 v_0$ as noted above, or in general the hyperbola $\{x = a_1\cosh u, y = b_1\sinh u\}$ at potential $U_0 v_1$, where $a_1 = a\cos v_1 - b\sin v_1$, $b_1 = a\sin v_1 + b\cos v_1$. This intersects the x-axis at $x_1 = a_1$. Since $a_1 = \sqrt{a^2+b^2}\cos(v_1 + atn(b/a))$, its most negative value gives the left-hand focus, $-\sqrt{a^2+b^2}$. The corresponding v_1 is $\pi - atn(b/a) = \pi/2 + atn(a/b)$, with $b_1 = 0$, representing a conducting sheet at potential $U_0\left[\frac{\pi}{2} + atn(a/b)\right]$ which extends from $x = -\sqrt{a^2+b^2}$ to $x = -\infty$. Some of these possibilities are illustrated in figure 2.

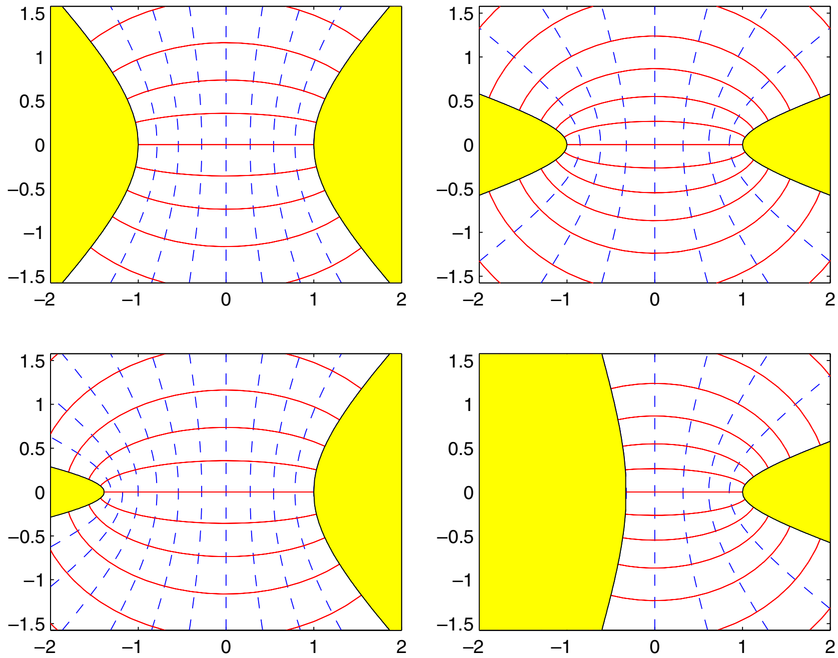

Figure 2. Four examples of configurations which are included in the results presented in this paper. The upper two plots have symmetric pairs of hyperbolic conductors, with $a = b$ on the left, $a = 3b$ on the right. In the lower two plots the right-hand conducting surfaces are unchanged, and are at zero potential. The lower-left plot has the left conductor at potential $(11/8)\Delta V$ (where ΔV is given in (10)), which puts its apex at about $-1.387a$, and makes its aspect ratio $|a'/b'| \approx 5.027$. The lower-right plot has the left conductor at potential $(5/8)\Delta V$; its apex is at about $-0.324a$, and its aspect ratio is ≈ 0.323.

3. Equations for the equipotentials and lines of force

From (7) written as $x = a' \cosh u$, $y = b' \sinh u$, it is clear that

$$\left(\frac{x}{a'}\right)^2 - \left(\frac{y}{b'}\right)^2 = 1 \tag{11}$$

which is the equation of a hyperbola. Since

$$a' = a\cos v - b\sin v \qquad b' = a\sin v + b\cos v \tag{12}$$

we see that $(a')^2 + (b')^2 = a^2 + b^2$, so the various equipotentials (corresponding to different values of $v = V/U_0$) are all confocal hyperbolae, with foci at $x = \pm\sqrt{a^2 + b^2}$.

To determine the equations of the lines of force, we eliminate the potential function v via $(a')^2 + (b')^2 = a^2 + b^2$: we get from (7) that

$$\left(\frac{x}{\cosh u}\right)^2 + \left(\frac{y}{\sinh u}\right)^2 = a^2 + b^2. \tag{13}$$

The field lines are thus ellipses, with semiaxes $\sqrt{a^2 + b^2}\cosh u$ and $\sqrt{a^2 + b^2}\sinh u$. The eccentricity is $e = \sqrt{1 - (\text{minor axis}/\text{major axis})^2} = \operatorname{sech} u$, the foci are at $\pm ae$. The central field line (at $y = 0$) corresponds to an ellipse of eccentricity 1, with foci at $x = \pm a$. Far from the axis the field lines approach circular arcs, since the eccentricity goes to zero exponentially for large $u = U/U_0$.

In the case of the flow of a fluid with negligible compressibility and viscosity, the roles are reversed: the equipotentials become the stream lines, and the lines of force become the curves of constant velocity potential.

4. The electric field strength

The electric field has components $E_x = -\partial_x V$, $E_y = -\partial_y V$, where $V = U_0 v$. The square of the field strength is thus

$$E^2 = U_0^2[(\partial_x v)^2 + (\partial_y v)^2]. \tag{14}$$

Now $w = u + \mathrm{i}v$ has its derivative with respect to $z = x + \mathrm{i}y$ equal to ([2], section 4.11)

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}u + \mathrm{i}\mathrm{d}v}{\mathrm{d}x + \mathrm{i}\mathrm{d}y} = \partial_x u - \mathrm{i}\partial_y u = \partial_y v + \mathrm{i}\partial_x v \tag{15}$$

since

$$\begin{aligned} \mathrm{d}u + \mathrm{i}\mathrm{d}v &= \partial_x u\,\mathrm{d}x + \partial_y u\,\mathrm{d}y + \mathrm{i}(\partial_x v\,\mathrm{d}x + \partial_y v\,\mathrm{d}y) \\ &= (\partial_x u - \mathrm{i}\partial_y u)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = (\partial_y v + \mathrm{i}\partial_x v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) \end{aligned}$$

from the Cauchy–Riemann equations. It follows that the electric field strength is given by $U_0\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|$ (this holds irrespective of whether we choose u or v to represent the potential). In the flow of an ideal fluid, the magnitude of the velocity is proportional to $|\mathrm{d}w/\mathrm{d}z|$.

In our case we have $z = a\cosh w + \mathrm{i}b\sinh w$, so

$$\frac{\mathrm{d}z}{\mathrm{d}w} = a\sinh w + \mathrm{i}b\cosh w = S(ac - bs) + \mathrm{i}C(as + bc) \tag{16}$$

where again $C = \cosh u$, $c = \cos v$, $S = \sinh u$ and $s = \sin v$. Thus

$$\left|\frac{\mathrm{d}z}{\mathrm{d}w}\right|^2 = (a^2 + b^2)C^2 - (ac - bs)^2 = (a^2 + b^2)C^2 - x^2/C^2 \tag{17}$$

and the square of the electric field is given by

$$E^2 = \frac{U_0^2}{(a^2 + b^2)C^2 - x^2/C^2}. \tag{18}$$

On the symmetry plane we have $u = 0$, so $C = 1$ and

$$E^2(y = 0) = \frac{U_0^2}{a^2 + b^2 - x^2}. \tag{19}$$

At the origin this is $U_0^2\big/(a^2 + b^2)$; at the right-hand hyperbolic conductor ($x = a$) the on-axis field is $U_0^2\big/b^2$. The ratio of these (and thus of the electrostrictive pressures) is $1 + a^2/b^2$.

To obtain the square of the field strength explicitly in terms of x and y, we note from (13) that $x^2/C^2 + y^2/S^2 = a^2 + b^2$; since $S^2 = C^2 - 1$ this gives us a quadratic for C^2, with solutions

$$C_\pm^2 = \frac{R^2 + r^2 \pm \sqrt{(R^2 + r^2)^2 - 4R^2x^2}}{2R^2} \qquad R^2 = a^2 + b^2, \quad r^2 = x^2 + y^2. \tag{20}$$

C_+ is the physical root. Hence (18) gives

$$E^2(x, y) = \frac{U_0^2}{R^2C_+^2 - x^2\big/C_+^2}. \tag{21}$$

On the symmetry plane ($y = 0$) this reduces to (19). On the $x = 0$ plane we have $C_+^2 = 1 + y^2/R^2$, so

$$E^2(x = 0) = \frac{U_0^2}{a^2 + b^2 + y^2}. \tag{22}$$

Figures 3 and 4 show the square of the field strength (or the square of the velocity in the fluid dynamics case) for pairs of hyperbolic conductors with $a = b$ and with $a = 2b$.

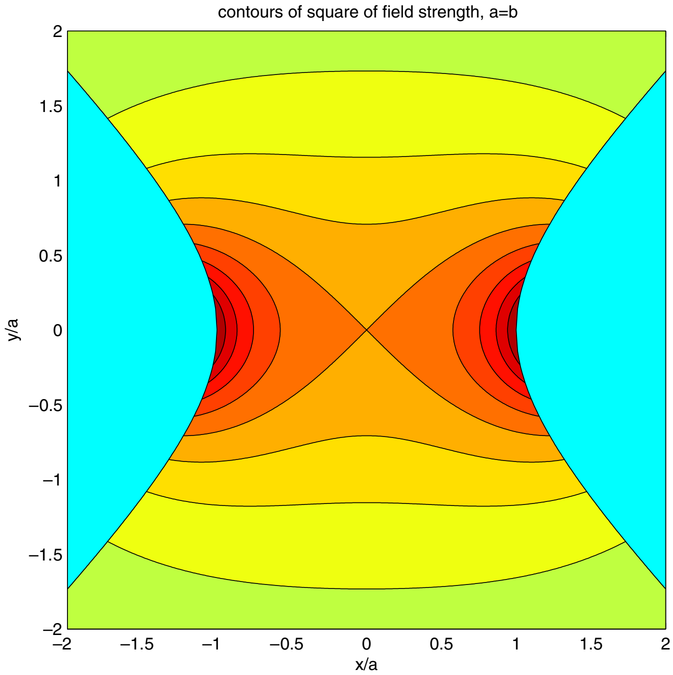

Figure 3. Contours of E^2, drawn for a pair of rectangular hyperbolae ($a = b$). The maximum field strength U_0/b occurs at the apex of each conductor. The contours are drawn for 90% of the maximum value of E^2, and then at 10% decrements. The value of E^2 at the centre is one-half of the maximum value U_0^2/b^2.

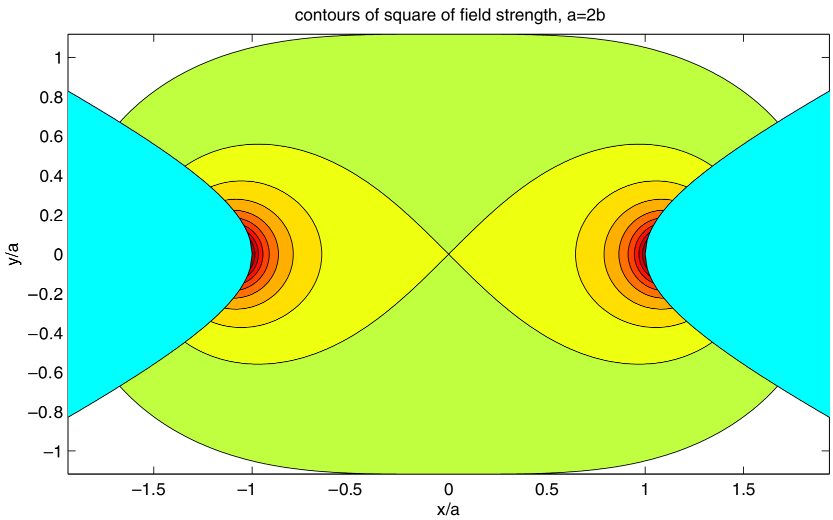

Figure 4. As for figure 3, the conductors being hyperbolae with $a = 2b$. In this case, the square of the field at the centre is one-fifth of its maximum value U_0^2/b^2 (in general the ratio is $b^2/(a^2+b^2)$). Comparison with figure 3 shows that the sharper conductors produce a more rapid decrease of E^2 (and thus of electrostrictive pressure) with distance from each tip.

5. The electric potential at a given point

The hyperbolae giving the equipotentials are given by (11) and (12). Since $(a')^2 + (b')^2 = a^2 + b^2 = R^2$, these equations give a quadratic for $(a')^2 = A$, namely

$$A^2 - (R^2 + r^2)A + R^2x^2 = 0 \tag{23}$$

with solutions

$$A_\pm = \tfrac{1}{2}\left\{R^2 + r^2 \pm \sqrt{(R^2 + r^2)^2 - 4R^2x^2}\right\}. \tag{24}$$

In this case A_- is the physical root, as can be seen for example by looking at the $y = 0$ plane, where $A = (a')^2 = x^2$. Also,

$$a' = a\cos v - b\sin v = R\cos(v + atn(b/a)). \tag{25}$$

Squaring and equating to A_- gives us

$$\begin{aligned} v(x, y) &= \frac{1}{2}acs\left(\frac{r^2 - \sqrt{(R^2 + r^2)^2 - 4R^2x^2}}{R^2}\right) - atn\left(\frac{b}{a}\right) \\ &= acs\left(\left[\frac{R^2 + r^2 - \sqrt{(R^2 + r^2)^2 - 4R^2x^2}}{2R^2}\right]^{\frac{1}{2}}\right) - atn\left(\frac{b}{a}\right). \end{aligned} \tag{26}$$

In the plane $y = 0$ these expressions reduce to

$$v(x, 0) = acs(x/R) - atn(b/a) \tag{27}$$

which is zero at $x = a$, and $\pi - 2atn(b/a) = 2atn(a/b)$ at $x = -a$. In the $x = 0$ plane, the potential function v takes the value $\pi/2 - atn(b/a) = atn(a/b)$.

6. Field strength and the curvature of conductor

There exists a pervasive misconception, discussed in detail in [7], that the strength of the electric field at a conducting surface is proportional to the local value of the curvature of the surface, $E = |\mathbf{E}| \sim \kappa$. This cannot be true in general, since $E^2 = (\partial_x V)^2 + (\partial_y V)^2$ depends on derivatives of the potential, which is a solution of Laplace's equation *and* of the boundary conditions. Thus E depends on the placement of other conductors, whereas κ is defined locally: for a surface $y(x)$,

$$\kappa = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}. \tag{28}$$

Price and Crowley [7] have shown that it is not even generally true that the maxima of E and κ occur together.

In the problem discussed here, however, the equipotentials all belong to a family of confocal hyperbolae, and we shall show that (in this case) there is a simple relationship between the field strength at the conducting surfaces and the local value of the curvature at the conductor. The relationship is not a linear one: the field strength is proportional to the cube root of the curvature.

We consider E and κ on the surface of the hyperbolic conductor $x^2/a^2 - y^2/b^2 = 1$. (The other conductor can be any one of the hyperbolic equipotential surfaces $x = a'\cosh u$, $y = b'\sinh u$ given in (7).) From $y^2 = b^2(x^2/a^2 - 1)$ we have $\mathrm{d}y/\mathrm{d}x = (x/y)(b^2/a^2)$ and $\mathrm{d}^2y/\mathrm{d}x^2 = -b^4/a^2y^3$, so

$$|\kappa| = \frac{a/b^2}{[1 + R^2y^2/b^4]^{\frac{3}{2}}} \qquad R^2 = a^2 + b^2. \tag{29}$$

On the $x^2/a^2 - y^2/b^2 = 1$ surface we have $C_+^2 = 1 + y^2/b^2$, so from (21) the electric field strength at this conductor is

$$E = \frac{U_0/b}{[1 + R^2y^2/b^4]^{\frac{1}{2}}}. \tag{30}$$

It follows that:

$$E = U_0\left(\frac{|\kappa|}{ab}\right)^{\frac{1}{3}}. \tag{31}$$

Thus there does exist a relationship, for the simple geometry being discussed here, between the field strength and the curvature, but it is not the expected linear one.

These formulae apply to any choice of hyperbolic equipotential surface as the conductor, with $a \to a'$ and $b \to b'$ where a' and b' are defined in (12). For a given conductor at potential $U_0 v$, the value of $(|\kappa|/a'b')^{\frac{1}{3}} U_0$ gives E. Since the conductor is an equipotential surface, a' and b' are constant on it, and E is proportional to the cube root of the local curvature. However, in the special case of the $x = 0$ plane we need the limit of $|\kappa|/a'$ as $a' \to 0$ to provide the y-dependence; κ itself is zero. In the plane $x = 0$ we have $a' \to 0$, $b' \to R$, which gives $|\kappa|/a' \to R^{-2}[1 + y^2/R^2]^{-\frac{3}{2}}$, so $E \to U_0(R^2 + y^2)^{-\frac{1}{2}}$, in agreement with (22).

7. Discussion

We have given analytic expressions for the electrostatic properties of confocal hyperbolic conductors. The same mathematics applies to the flow of incompressible nonviscous fluid between hyperbolic barriers. These two features should be useful in the design and analysis of the proposed Packard experiment [1], and also in modelling fields in electric force microscopy (see for example [8]).

The results are simple enough to be used as an example in teaching electrostatics or fluid mechanics. Particularly interesting is the fact that there exists a power relationship between the field strength at the surface of a conductor and the local curvature, for this family of conductor surfaces.

Acknowledgments

The author is grateful to Professor Richard Packard for asking 'what would be the electric field between two razor blades, as a function of the radius of curvature of the blade edges and of their separation?', and for stimulating conversations about his proposed experiment.

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