

## Acoustic beam invariants

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Transversely finite sound beams traveling in fluid media with negligible viscous damping and scattering have seven quantities constant along the length of the beam. The simplest of these is the cycle-averaged momentum per unit length of the beam. The seven constant quantities are proved to be invariants from conservation laws. An angular momentum flux density tensor is introduced, in the formulation of conservation of angular momentum. Examples of the invariants are given, for approximate solutions of the wave equation (Gaussian beams), and for a set of exact solutions (generalized Bessel beams).

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### I. INTRODUCTION

Electromagnetic beams and particle beams each have seven invariants associated with conservation laws [1,2]. By “invariants” we mean quantities that do not change along the length of the beam, as the beam diverges from (or converges to) a focal region. This paper establishes the existence of analogous invariants for sound beams.

Figure 1 shows the energy and momentum densities of the  $\psi_{11}^G$  sound beam discussed in Sec. V, in its focal region. The momentum density (indicated by arrows) varies strongly through the focal region, yet its integral over a transverse section does not change. This integral is one of the invariants to be established below.

It has recently been shown [3] that the cycle-averaged energy, momentum, and angular momentum per unit length of an acoustic beam are all constant for beams derived from a class of exact solutions of the wave equation. If the beam propagates in the  $z$  direction, these quantities are

$$E' = \int d^2r \bar{e}, \quad P'_z = \int d^2r \bar{p}_z, \quad J'_z = \int d^2r \bar{j}_z, \quad (1)$$

where  $\int d^2r = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy = \int_0^{\infty} dr \int_0^{2\pi} d\phi$  in Cartesian and cylindrical polar coordinates, respectively. The quantities  $e$ ,  $p_z$ , and  $j_z$  are densities:  $e(\mathbf{r}, t)$  is the energy per unit volume,  $p_z(\mathbf{r}, t)$  and  $j_z(\mathbf{r}, t)$  are, respectively, the  $z$  components of the momentum and angular momentum per unit volume. Overbars denote cycle averaging, e.g.,  $\bar{e}(\mathbf{r}) = T^{-1} \int_0^T dt e(\mathbf{r}, t)$ , where  $T = 2\pi/\omega$  is the period everywhere in the beam, assumed to be of one frequency. The energy content in a slice  $dz$  of the beam is  $dE = E' dz$ ; hence our notation  $E'$ , which equals  $dE/dz$ .

One might think that the constancy of  $E'$ ,  $P'_z$ , and  $J'_z$  follows immediately from the conservation of energy, momentum, and angular momentum, respectively. This is not so: in fact,  $P'_z$  is truly an invariant (always independent of  $z$ , for any sound beam corresponding to an exact solution of the wave equation), while  $E'$  and  $J'_z$  are not invariants in general. Further, the invariance of  $P'_z$  follows from the continuity equation (conservation of matter), not from momentum conservation. We shall prove that three invariants follow from the conservation of momentum, and three more from the conservation of angular momentum. These invariants are integrals over elements of the momentum flux density tensor

and of the angular momentum flux density tensor, respectively.

There is recent interest in acoustical tweezers [4,5], and there may in the future be applications of acoustical spanners (using helicoidal beams [6–10]). Momentum and angular momentum flux densities are needed in the calculation of forces and torques, and thus underpin the requisite theory of acoustical tweezers and spanners, respectively.

### II. CONSERVATION OF MATTER

The continuity equation [11]

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2)$$

has the cycle average  $\nabla \cdot \bar{\mathbf{p}} = 0$ , where  $\mathbf{p} = \rho \mathbf{v}$  is the momentum density. Operating with  $\int d^2r$  on  $\nabla \cdot \bar{\mathbf{p}} = 0$  gives

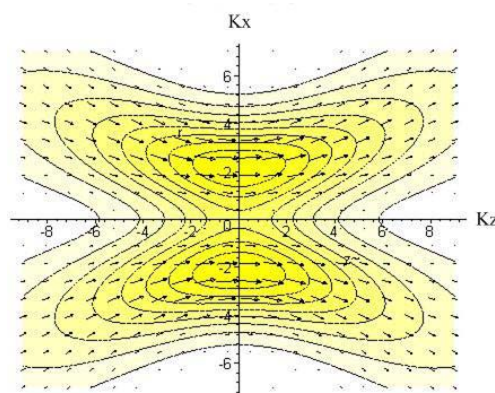


FIG. 1. (Color online) The cycle-averaged energy density (contours) and momentum density (arrows) for the  $\psi_{11}^G$  sound beam of Sec. V, plotted for  $Kb=5$ . The beam is hollow in momentum, because of its helicoidal nature. A three-dimensional picture would be obtained by rotating the figure about the horizontal axis (the  $z$  axis). The azimuthal component of the momentum is not shown. The invariants of this paper are integrals over a transverse section of the beam at constant  $z$ , of physical quantities such as the longitudinal component of the momentum density  $p_z$ .

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \{ \partial_x \bar{p}_x + \partial_y \bar{p}_y + \partial_z \bar{p}_z \} = 0. \quad (3)$$

We consider acoustic beams with net momentum in the  $z$  direction, and transversely finite in the  $x$  and  $y$  directions. For such beams the first two terms in Eq. (3) are zero. For example,  $\int_{-\infty}^{\infty} dx \partial_x \bar{p}_x = \bar{p}_x|_{-\infty}^{\infty} = 0$ . We are left with

$$\partial_z \int d^2r \bar{p}_z = \partial_z P'_z = 0, \quad (4)$$

i.e., the momentum content per unit length of the beam  $P'_z$  does not change along the length of the beam. This is our first invariant, derived without approximation from the conservation of matter. Note that the proof of invariance of the momentum per unit length is based on conservation of matter, not of momentum.

The momentum density has first, second, and higher-order terms

$$\begin{aligned} \mathbf{p} &= \rho \mathbf{v} = (\rho_0 + \rho_1 + \rho_2 + \dots)(\mathbf{v}_1 + \mathbf{v}_2 + \dots) \\ &= \rho_0 \mathbf{v}_1 + (\rho_1 \mathbf{v}_1 + \rho_0 \mathbf{v}_2) + \dots \end{aligned} \quad (5)$$

The first-order velocity  $\mathbf{v}_1$  has zero curl, and can thus be written as the gradient of a velocity potential  $V(\mathbf{r}, t)$ ,  $\mathbf{v}_1 = \nabla V$ . The velocity potential  $V$  satisfies the wave equation [11]. For acoustic beams of angular frequency  $\omega$  we shall work in terms of a complex potential

$$\Psi(\mathbf{r}, t) = e^{-i\omega t} \psi(\mathbf{r}), \quad V(\mathbf{r}, t) = \text{Re } \Psi \text{ or } \text{Im } \Psi, \quad (6)$$

where the spatial part  $\psi(\mathbf{r})$  satisfies the Helmholtz equation

$$(\nabla^2 + K^2)\psi(\mathbf{r}) = 0, \quad K = \omega/c. \quad (7)$$

It is clear that the cycle-averages of  $V$  and its derivatives are zero, so  $\bar{\mathbf{v}}_1 = 0$ . We have previously shown (Ref. [3], Sec. III) that  $\bar{\mathbf{v}}_2 = 0$  also. Thus the lowest non-zero term in the cycle-average of the momentum density is [3]

$$\overline{\rho_1 \mathbf{v}_1} = -\rho_0 c^{-2} \overline{(\partial_t V)(\nabla V)} = \frac{K\rho_0}{2c} \text{Im}(\psi^* \nabla \psi). \quad (8)$$

The final expression holds for both  $V = \text{Re } \Psi$  and  $V = \text{Im } \Psi$ . Thus

$$P'_z = \frac{K\rho_0}{2c} \int d^2r \text{Im}(\psi^* \partial_z \psi). \quad (9)$$

We have written an expression for this invariant as an exact equality, it being understood here (and for like expressions in the following) that it is exact only to second-order in the velocity potential.

The invariance of  $P'_z$  also follows from the conservation of energy, as we shall now show. The energy density of a fluid in which there propagates an acoustic field is [12]

$$e(\mathbf{r}, t) = e_0 + (e_0 + p_0)\rho_0^{-1}\rho_1 + \frac{1}{2}c^2\rho_0^{-1}\rho_1^2 + \frac{1}{2}\rho_0 v_1^2 + \dots, \quad (10)$$

where  $\rho_0$ ,  $e_0$ , and  $p_0$  are the density, energy density, and pressure in the undisturbed fluid. The first-order density is given by [11]

$$\rho_1 = -\rho_0 c^{-2} \partial_t V. \quad (11)$$

The first-order term in Eq. (10) is  $e_1 = (e_0 + p_0)\rho_0^{-1}\rho_1$ ; using Eq. (11) and the fact that  $V$  satisfies the wave equation gives us

$$\partial_t e_1 + \nabla \cdot \mathbf{S}_1 = 0, \quad \mathbf{S}_1 = (e_0 + p_0)\mathbf{v}_1, \quad (12)$$

where  $\mathbf{S}_1$  is the first-order energy flux density. Since the cycle-average of  $\mathbf{v}_1$  is zero, Eq. (12) cycle averages to zero.

Moving on to the second-order energy density term  $e_2 = \frac{1}{2}c^2\rho_0^{-1}\rho_1^2 + \frac{1}{2}\rho_0 v_1^2$ , and again using  $\partial_{ct}^2 V = \nabla^2 V$ , we find after some rearrangement that conservation of energy in second order takes the form

$$\partial_t e_2 + \nabla \cdot \mathbf{S}_2 = 0, \quad \mathbf{S}_2 = -\rho_0 (\partial_t V) \nabla V. \quad (13)$$

The cycle-average of Eq. (13) is  $\nabla \cdot \bar{\mathbf{S}}_2 = 0$ , leading to the invariant  $\int d^2r \bar{S}_{2z}$ . Apart from the factor  $c^2$ , this is the same as the invariant we have previously derived from matter conservation [compare Eq. (8)]. An analogous situation pertains for electromagnetic beams [1], where the momentum per unit length invariant arises from energy conservation.

### III. CONSERVATION OF MOMENTUM

Landau and Lifshitz [11] show from the continuity equation (2) and from the Euler equation that conservation of momentum in an isotropic inviscid fluid can be expressed as

$$\partial_t(\rho v_i) + \sum_k \partial_k \Pi_{ik} = 0 \quad (i, k = x, y, z), \quad (14)$$

where the momentum flux density tensor is given by

$$\Pi_{ik} = (p - p_0)\delta_{ik} + \rho v_i v_k. \quad (15)$$

[We have subtracted the constant diagonal tensor  $p_0\delta_{ik}$  from the Landau-Lifshitz expression, as we can do since only derivatives of  $\Pi_{ik}$  occur in Eq. (14). Again  $p_0$  is the pressure in the undisturbed fluid. With our definition, the zero-order term  $\Pi_{ik}^{(0)}$  is identically zero.]

The time average of Eq. (14) gives  $\sum_k \partial_k \bar{\Pi}_{ik} = 0$ , and integration over the  $xy$  plane gives  $\partial_z \int d^2r \bar{\Pi}_{iz} = 0$ . Thus the integrals  $\int d^2r \bar{\Pi}_{iz}$  are invariants, to all orders in the velocity potential.

The first-order term in Eq. (14) is

$$\rho_0 \partial_t v_i - \rho_0 \sum_k \delta_{ik} \partial_k \partial_t V = 0 \quad (16)$$

since the first-order pressure term is given by [12]

$$p_1 = c^2 \rho_1 = -\rho_0 \partial_t V. \quad (17)$$

The cycle-average of both terms in Eq. (16) is zero. We are thus left with the second-order terms in Eq. (14). The cycle-

average of the time derivative is zero to any order; what remains is

$$0 = \sum_k \partial_k \overline{\Pi_{ik}^{(2)}} = \sum_k \partial_k (\overline{p_2 \delta_{ik}} + \rho_0 \overline{v_i v_k}), \quad (18)$$

where  $p_2$  is the second-order pressure, and is given by [12]

$$p_2 = c^2(\rho_2 + \alpha \rho_0^{-1} \rho_1^2), \quad \alpha = \frac{\rho_0}{2c^2} \left( \frac{\partial c^2}{\partial \rho} \right)_0. \quad (19)$$

(The subscript zero indicates that the derivative is to be evaluated adiabatically for the fluid at equilibrium.) The second-order density term  $\rho_2$  satisfies an inhomogeneous wave equation [12], with the source term second-order in the velocity potential  $V$ . We set  $\rho_2 = \rho_0 c^{-2} R$ ; from Eq. (A3) of Ref. [12],

$$(\partial_{ct}^2 - \nabla^2)R = -K^2 \nabla \cdot (V \nabla V) + \frac{1}{2} \nabla^2 (\nabla V)^2 + \alpha \nabla^2 (\partial_{ct} V)^2, \quad (20)$$

where we have used the fact that the velocity potential satisfies the wave equation. We set  $V(\mathbf{r}, t) = C(\mathbf{r}) \cos \omega t + S(\mathbf{r}) \sin \omega t$  and cycle average (20), using the fact that  $C$  and  $S$  satisfy the Helmholtz equation (7). We find  $\overline{V \nabla V} = \frac{1}{2} \nabla \overline{V^2}$  and  $\overline{(\partial_{ct} V)^2} = K^2 \overline{V^2}$ , so that

$$-\nabla^2 \overline{R} = K^2 \left( \alpha - \frac{1}{2} \right) \nabla^2 \overline{V^2} + \frac{1}{2} \nabla^2 (\overline{\nabla V})^2. \quad (21)$$

This equation is solved by

$$\overline{R} = R_0 - K^2 \left( \alpha - \frac{1}{2} \right) \overline{V^2} + \frac{1}{2} \overline{(\nabla V)^2}, \quad (22)$$

where  $R_0$  is any solution of  $\nabla^2 \overline{R} = 0$ . Since  $R_0$  is a harmonic function, it can have extrema only at domain boundaries. In the case of a beam of finite transverse extent in an unbounded fluid,  $R_0$  has to be constant in space, and this constant has to be zero.

In addition to  $\overline{\rho_2} = \rho_0 c^{-2} \overline{R}$ , we also need

$$\overline{\rho_1^2} = \rho_0^2 c^{-4} \overline{(\partial_t V)^2} = \rho_0^2 c^{-2} K^2 \overline{V^2}. \quad (23)$$

Thus the second-order pressure term cycle averages to

$$\overline{p_2} = c^2 (\overline{\rho_2} + \alpha \rho_0^{-1} \overline{\rho_1^2}) = \rho_0 (\overline{R} + \alpha K^2 \overline{V^2}) = \frac{1}{2} \rho_0 [K^2 \overline{V^2} - \overline{(\nabla V)^2}]. \quad (24)$$

Note that the term depending on  $\alpha$  has cancelled out. The cycle average of the second-order part of the momentum flux density tensor is therefore

$$\overline{\Pi_{ik}^{(2)}} = \rho_0 \left\{ \frac{1}{2} [K^2 \overline{V^2} - \overline{(\nabla V)^2}] \delta_{ik} + \overline{(\partial_i V)(\partial_k V)} \right\}. \quad (25)$$

By operating on Eq. (18) with  $\int d^2 r$ , we obtain three invariants associated with the conservation of momentum:

$$P'_{xz} = \int d^2 r \overline{\Pi_{xz}^{(2)}}, \quad P'_{yz} = \int d^2 r \overline{\Pi_{yz}^{(2)}}, \quad P'_{zz} = \int d^2 r \overline{\Pi_{zz}^{(2)}}. \quad (26)$$

Whether the velocity potential  $V(\mathbf{r}, t)$  is expressed as the real or the imaginary part of the complex potential  $\Psi = e^{-i\omega t} \psi(\mathbf{r})$ , we have

$$\overline{V^2} = \frac{1}{2} |\psi|^2, \quad \overline{(\partial_x V)^2} = \frac{1}{2} |\partial_x \psi|^2, \quad \overline{(\nabla V)^2} = \frac{1}{2} |\nabla \psi|^2$$

$$\overline{(\partial_x V)(\partial_z V)} = \frac{1}{2} \text{Re}[(\partial_x \psi)(\partial_z \psi^*)] \quad (27)$$

(and parallel results involving derivatives with respect to  $y$ ). Thus

$$P'_{zz} = \frac{1}{2} \rho_0 \int d^2 r \{ K^2 \overline{V^2} + \overline{(\partial_z V)^2} - \overline{(\partial_x V)^2} - \overline{(\partial_y V)^2} \}$$

$$= \frac{1}{4} \rho_0 \int d^2 r \{ K^2 |\psi|^2 + |\partial_z \psi|^2 - |\partial_r \psi|^2 - r^{-2} |\partial_\phi \psi|^2 \}. \quad (28)$$

To obtain the last expression we have used Eq. (27) and also changed to cylindrical polar coordinates  $r, \phi, z$ .

The off-diagonal invariants  $P'_{xz}$  and  $P'_{yz}$  will often be zero by symmetry. This is clear for the case where  $\psi$  is independent of  $\phi$ , since  $\partial_x = \cos \phi \partial_r - r^{-1} \sin \phi \partial_\phi$ , so, for example,

$$P'_{xz} = \rho_0 \int_0^\infty dr r \int_0^{2\pi} d\phi \frac{1}{2} \text{Re}[(\partial_x \psi)(\partial_z \psi^*)] \quad (29)$$

integrates to zero over  $\phi$ . The same is true when the azimuthal dependence of  $\psi$  is purely in the factor  $e^{im\phi}$ , since only terms linear in  $\cos \phi$  and  $\sin \phi$  remain in the integrand of Eq. (29).

#### IV. CONSERVATION OF ANGULAR MOMENTUM

There are three more invariants of acoustic beams, arising from the conservation of angular momentum. As in the electromagnetic case [1], we need an angular momentum flux density tensor. This is formed from the momentum flux density tensor, the Cartesian coordinates, and the completely antisymmetric tensor  $\varepsilon_{ijk}$  with  $\pm 1$  or zero as elements ( $\varepsilon_{123} = 1$ ,  $\varepsilon_{132} = -1$ ,  $\varepsilon_{113} = 0$ , etc):

$$\Lambda_{li} = \sum_j \sum_k \varepsilon_{ijk} x_j \Pi_{kl}. \quad (30)$$

Then the rate of change of the angular momentum density  $\mathbf{j}(\mathbf{r}, t)$  is given by

$$\partial_t j_i + \sum_l \partial_l \Lambda_{li} = 0. \quad (31)$$

This equation expresses the conservation of angular momentum in an acoustic beam. The exact invariants which follow from the conservation law (31) are  $\int d^2 r \overline{\Lambda_{zi}}$  ( $i = x, y, z$ ).

As for the momentum flux density tensor, the lowest non-zero cycle-average is of second order:

$$\overline{\Lambda_{li}^{(2)}} = \sum_j \sum_k \varepsilon_{ijk} x_j \overline{\Pi_{kl}^{(2)}}. \quad (32)$$

The invariants arise, as before, from integrating the cycle-averaged angular momentum conservation equation (31) over a slice of the sound beam at constant  $z$ . These invariants are

$$L'_{zi} = \int d^2r \overline{\Lambda_{zi}^{(2)}} \quad (i = x, y, z). \quad (33)$$

Explicitly, the three invariants arising from the conservation of angular momentum are

$$\begin{aligned} L'_{zx} &= \int d^2r [y \overline{\Pi_{zz}^{(2)}} - z \overline{\Pi_{yz}^{(2)}}], \\ L'_{zy} &= \int d^2r [z \overline{\Pi_{xz}^{(2)}} - x \overline{\Pi_{zz}^{(2)}}], \\ L'_{zz} &= \int d^2r [x \overline{\Pi_{yz}^{(2)}} - y \overline{\Pi_{xz}^{(2)}}]. \end{aligned} \quad (34)$$

When the azimuthal dependence in  $\psi$  is absent or purely in the factor  $e^{im\phi}$ ,  $L'_{zx}$  and  $L'_{zy}$  will be zero, by arguments parallel to those used for  $P'_{xz}$  and  $P'_{yz}$  above.

The  $L'_{zz}$  invariant simplifies considerably when expressed in cylindrical polar coordinates: from

$$\begin{aligned} \overline{\Pi_{xz}^{(2)}} &= \rho_0 \overline{(\partial_x V) \partial_z V} = \frac{1}{2} \rho_0 \operatorname{Re}\{(\partial_x \psi)(\partial_z \psi^*)\} \\ &= \frac{1}{2} \rho_0 \operatorname{Re}\left\{\left(\cos \phi \partial_r \psi - \frac{\sin \phi}{r} \partial_\phi \psi\right)(\partial_z \psi^*)\right\} \end{aligned} \quad (35)$$

and

$$\overline{\Pi_{yz}^{(2)}} = \frac{1}{2} \rho_0 \operatorname{Re}\left\{\left(\sin \phi \partial_r \psi + \frac{\cos \phi}{r} \partial_\phi \psi\right)(\partial_z \psi^*)\right\} \quad (36)$$

we have

$$x \overline{\Pi_{yz}^{(2)}} - y \overline{\Pi_{xz}^{(2)}} = \frac{1}{2} \rho_0 \operatorname{Re}\{(\partial_\phi \psi)(\partial_z \psi^*)\} \quad (37)$$

so that

$$L'_{zz} = \frac{1}{2} \rho_0 \int d^2r \operatorname{Re}\{(\partial_\phi \psi)(\partial_z \psi^*)\}. \quad (38)$$

## V. EXAMPLE A: PARAXIAL GAUSSIAN BEAMS

Gaussian beams are approximate solutions of the Helmholtz equation; the simplest example is the ‘‘fundamental Gaussian mode’’

$$\psi_{00}^G = \frac{b}{b + iz} \exp\left\{iKz - \frac{Kr^2}{2(b + iz)}\right\}. \quad (39)$$

Here  $b$  is the Rayleigh range: the beam waist extends longitudinally over a range of about  $2b$ . A measure of the beam

waist transverse size is  $(2b/K)^{1/2}$ . Far from the waist the beam diverges at angle  $\arctan(2/Kb)^{1/2}$  to the axis. When  $\psi_{00}^G$  is substituted into the Helmholtz equation (7), we find

$$\begin{aligned} K^{-2}(\psi_{00}^G)^{-1}(\nabla^2 + K^2)\psi_{00}^G &= \frac{-2}{K^2(b + iz)^2} + \frac{2r^2}{K(b + iz)^3} \\ &\quad - \frac{r^4}{4(b + iz)^4} \end{aligned} \quad (40)$$

instead of zero. The right-hand side of Eq. (40) is negligible in the region where both of the following relations hold true:

$$K^2(b^2 + z^2) \gg 1 \quad \text{and} \quad b^2 + z^2 \gg r^2. \quad (41)$$

The left-hand sides of both inequalities are smallest in the focal plane  $z=0$ . The Gaussian beam  $\psi_{00}^G$  is thus a good approximate solution of the Helmholtz equation when  $\beta = Kb \gg 1$  and  $b \gg r$  (or  $\beta \gg Kr$ ).

We shall calculate the quantities  $E'$ ,  $cP'_z$ ,  $\omega J'_z$ ,  $P'_{zz}$ , and  $KL'_{zz}$ , for the fundamental  $\psi_{00}^G$  and for a higher mode  $\psi_{11}^G$  to be defined below. These quantities all have the same dimension (of energy per unit length), and from the discussion above we expect meaningful results for Gaussian beams when  $\beta \gg 1$ . Let us first define  $\psi_{11}^G$ :

$$\psi_{11}^G = \frac{x + iy}{b + iz} \psi_{00}^G = \frac{r e^{i\phi}}{b + iz} \psi_{00}^G. \quad (42)$$

For  $\psi_{11}^G$  the same operation as in Eq. (40) gives the right-hand side

$$\frac{-6}{K^2(b + iz)^2} + \frac{3r^2}{K(b + iz)^3} - \frac{r^4}{3(b + iz)^4}. \quad (43)$$

This will again be small at all  $z$  when  $\beta = Kb$  is large compared to unity and to  $Kr$ .

The quantities we wish to calculate are the invariants  $cP'_z$ ,  $P'_{zz}$ , and  $KL'_{zz}$ , and the (in general) noninvariant quantities  $E'$  and  $\omega J'_z$ . The invariants are, from Eqs. (9), (28), and (38),

$$cP'_z = \frac{1}{2} K \rho_0 \int d^2r \operatorname{Im}(\psi^* \partial_z \psi),$$

$$P'_{zz} = \frac{1}{4} \rho_0 \int d^2r \{K^2 |\psi|^2 + |\partial_z \psi|^2 - |\partial_r \psi|^2 - r^{-2} |\partial_\phi \psi|^2\},$$

$$KL'_{zz} = \frac{1}{2} K \rho_0 \int d^2r \operatorname{Re}\{(\partial_\phi \psi)(\partial_z \psi^*)\}. \quad (44)$$

The noninvariants giving energy and angular momentum content per unit length are given by [3]

$$E' = \frac{1}{4} \rho_0 \int d^2r [|\nabla \psi|^2 + K^2 |\psi|^2],$$

$$\omega J'_z = \frac{1}{2} K^2 \rho_0 \int d^2r \operatorname{Im}(\psi^* \partial_\phi \psi). \quad (45)$$

When  $\psi$  is set equal to  $V_0 \psi_{00}^G$  ( $V_0$  is the amplitude of the velocity potential  $V$ ), lack of azimuthal dependence auto-

matically makes  $L'_{zz}$  and  $J'_z$  zero. The nonzero quantities for the fundamental Gaussian beam  $\psi_{00}^G$  are

$$\begin{aligned} cP'_z &= \frac{\pi}{2}\rho_0 V_0^2 \left( \beta - \frac{1}{2} \right), \\ E' &= \frac{\pi}{2}\rho_0 V_0^2 \left( \beta + \frac{1}{4}\beta^{-1} \right), \\ P'_{zz} &= \frac{\pi}{2}\rho_0 V_0^2 \left( \beta - 1 + \frac{1}{4}\beta^{-1} \right). \end{aligned} \quad (46)$$

Of these  $cP'_z$  and  $P'_{zz}$  must be independent of  $z$ , as we have proved. The fact that  $E'$  is also constant along the length of the beam is interesting, but, as we noted above, the Gaussian wave function is approximate and the results derived from it can only be expected to be accurate in the limit of large  $\beta = Kb$ .

The  $\psi_{11}^G$  beam carries angular momentum, so we have more nonzero quantities: the nonzero universal invariants are (with  $\psi$  set equal to  $V_0\psi_{11}^G$ )

$$\begin{aligned} cP'_z &= \frac{\pi}{2}\rho_0 V_0^2 (1 - \beta^{-1}), \\ P'_{zz} &= \frac{\pi}{2}\rho_0 V_0^2 \left( 1 - 2\beta^{-1} + \frac{3}{4}\beta^{-2} \right), \\ KL'_{zz} &= \frac{\pi}{2}\rho_0 V_0^2 (1 - \beta^{-1}). \end{aligned} \quad (47)$$

For the energy and angular momentum contents per unit length of the beam we find

$$\begin{aligned} E' &= \frac{\pi}{2}\rho_0 V_0^2 \left( 1 + \frac{3}{4}\beta^{-2} \right), \\ \omega J'_z &= \frac{\pi}{2}\rho_0 V_0^2. \end{aligned} \quad (48)$$

Again  $E'$  and  $J'_z$  are independent of  $z$ . The remarks about the validity of the Gaussian wave functions made below Eq. (46) apply here equally.

## VI. EXAMPLE B: GENERALIZED BESSEL BEAMS

A set of exact solutions to the wave equation were discussed in the context of angular-momentum carrying acoustic beams in Ref. [3]. These solutions are

$$\psi_m(\mathbf{r}) = e^{im\phi} \int_0^K dk f(k) e^{iqz} J_m(kr), \quad k^2 + q^2 = K^2. \quad (49)$$

We have already evaluated the energy, momentum, and angular momentum per unit length in Ref. [3], Eqs. (26), (22), and (21), respectively. It remains for us to evaluate  $P'_{zz}$  and  $L'_{zz}$ . Since  $\partial_\phi \psi_m = im\psi_m$ , we have from Eq. (44)

$$P'_{zz} = \frac{1}{4}\rho_0 \int d^2r \left\{ K^2 |\psi|^2 + |\partial_z \psi|^2 - |\partial_r \psi|^2 - \frac{m^2}{r^2} |\psi|^2 \right\}. \quad (50)$$

The  $r$  integration of the  $|\partial_r \psi|^2$  term gives

$$\begin{aligned} &\int_0^\infty dr r [\partial_r J_m(kr)] [\partial_r J_m(k'r)] \\ &= \int_0^\infty dr r \left( k^2 - \frac{m^2}{r^2} \right) J_m(kr) J_m(k'r) \end{aligned} \quad (51)$$

by integration by parts, and use of the Bessel differential equation. Hence the  $m^2/r^2$  terms in Eq. (51) cancel, and we are left with

$$\begin{aligned} P'_{zz} &= 2\pi \frac{\rho_0}{4} \int_0^\infty dr r \int_0^K dk f^*(k) \int_0^K dk' f(k') e^{i(q'-q)z} \\ &\quad \times \{K^2 + qq' - k^2\} J_m(kr) J_m(k'r). \end{aligned} \quad (52)$$

We now apply Hankel's integral formula [13], in the form [14]

$$\int_0^\infty dr r J_m(kr) J_m(k'r) = k^{-1} \delta(k - k'). \quad (53)$$

This reduces Eq. (52) to

$$P'_{zz} = \pi \rho_0 \int_0^K dk k^{-1} q^2 |f(k)|^2. \quad (54)$$

Finally, we evaluate the nonzero invariant related to angular momentum conservation. This is, from Eq. (44) with  $\partial_\phi \psi = im\psi$ ,

$$\begin{aligned} L'_{zz} &= \pi \rho_0 \int_0^\infty dr r \operatorname{Re} \{ im \psi \partial_z \psi^* \} \\ &= \pi \rho_0 \operatorname{Re} \left\{ im \int_0^\infty dr r \int_0^K dk f(k) e^{iqz} J_m(kr) \right. \\ &\quad \left. \times \int_0^K dk' f^*(k') (-iq') e^{-iq'z} J_m(k'r) \right\} \\ &= m \pi \rho_0 \int_0^K dk k^{-1} q |f(k)|^2 \end{aligned} \quad (55)$$

again by use of Hankel's integral formula.

We summarize the results by listing the multiplier  $M$  in the generic expression

$$\pi \rho_0 \int_0^K dk k^{-1} |f(k)|^2 M. \quad (56)$$

The multipliers for the three nonzero universal invariants are

$$cP'_z: Kq, \quad P'_{zz}: q^2, \quad KL'_{zz}: mKq. \quad (57)$$

For the quantities  $E'$  and  $\omega J'_z$ , the multipliers are

$$E':K^2, \quad \omega J'_z:mK^2. \quad (58)$$

We note the factor  $m$  (the “topological charge” of the beam) between  $cP'_z$  and  $KL'_{zz}$ , and between  $E'$  and  $\omega J'_z$ . This is in accord with the view that the acoustic beam consists of phonons, each of energy  $\hbar\omega$ , momentum along the beam axis  $\hbar K = \hbar\omega/c$ , and angular momentum along the beam axis  $\hbar m$ .

## VII. DISCUSSION

We have shown that, in general, there are seven invariants of single-frequency acoustic beams, arising from conservation laws. For both the generalized Bessel beams, and for the more limited examples of paraxial Gaussian beams, the results are consistent with the phonon picture. Note, however,

that for phonons the energy is  $c$  times the  $z$  component of the momentum, whereas the cycle-averaged energy density is greater than  $c$  times the cycle-averaged  $z$  component of the momentum density [3].

In the case of beams based on solutions of the wave equation with  $e^{im\phi}$  azimuthal dependence, there are in general only three nonzero invariants. It is both an attractive feature and a puzzle that the generalized Bessel beams have two more quantities which do not change along the length of the beam: the energy content per unit length  $E'$ , and the angular momentum content per unit length  $J'_z$ . The special feature which may be responsible is the absence of evanescent waves in Eq. (49), i.e., those with imaginary  $q$ . This comes from limiting the integration over  $k$  to the range 0 to  $K$ . An analogous situation pertains in the electromagnetic case [14].

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