Journal of Optics

PAPER

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To cite this article: John Lekner 2019 *J. Opt.* **21** 035402

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IOP Publishing Journal of Optics

J. Opt. **21** (2019) 035402 (7pp) https://doi.org/10.1088/2040-8986/ab026f

Chirality of self-dual electromagnetic beams

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Received 6 August 2018, revised 25 December 2018
Accepted for publication 28 January 2019
Published 14 February 2019

Abstract

Self-dual electromagnetic fields are those unchanged by the substitutions $\boldsymbol{E} \to \boldsymbol{B}$, $\boldsymbol{B} \to -\boldsymbol{E}$. We show that the chiral density and the chiral current in self-dual monochromatic beams are proportional to the energy and momentum densities, respectively. It is also shown that *self-duality of monochromatic fields implies maximal chirality*. This property follows from the fact that self-dual monochromatic electric and magnetic fields are eigenvectors of the curl operator.

Keywords: chirality, self-dual beams, curl

(Some figures may appear in colour only in the online journal)

1. Introduction

The chirality of matter, molecules and crystals, leads to optical activity or rotatory power, the ability of a medium to rotate the plane of polarization of light. Its discovery dates back to work by Arago in 1811, Biot (five memoirs between 1812 and 1837) and Fresnel's 1822 conjecture that on entering an optically active medium light is split into two beams of opposite circular polarization which travel with different phase velocities. In 1848 Pasteur demonstrated that the rotatory power of a tartrate solution is related to the form the tartrate crystals take: crystals of opposite handedness dissolve to give solutions with opposite rotatory power [1].

Here we are concerned with the chirality of light itself, and with the special properties of *self-dual* light beams and of their chiral measures. The free-space Maxwell *equations* are unchanged by the duality transformation $\boldsymbol{E} \to \boldsymbol{B}$, $\boldsymbol{B} \to -\boldsymbol{E}$. However, *solutions* of the Maxwell equations are in general changed by the duality transformation into physically different solutions. For example, a transverse electric (TE) beam is changed into a transverse magnetic (TM) beam. The author has previously explored properties of self-dual electromagnetic beams (those unchanged by the duality transformation). The first of these were the TM $\pm$ iTE beams [2, 3]; the notation is a shorthand for the superposition of the fields of a TM beam and of a TE beam, in phase quadrature.

Recent interest in the chiral properties of electromagnetic fields has interpreted Lipkin's conserved quantities as the chiral density χ and chiral current $\boldsymbol{C}$, defined for real fields $\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{B}(\boldsymbol{r}, t)$ as [4–16]

$$\chi = \boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) + \boldsymbol{B} \cdot (\nabla \times \boldsymbol{B}), \tag{1.1}$$

$$\boldsymbol{C} = \boldsymbol{E} \times (\nabla \times \boldsymbol{B}) - \boldsymbol{B} \times (\nabla \times \boldsymbol{E}). \tag{1.2}$$

The geometric meaning of terms such $\boldsymbol{E} \cdot (\nabla \times \boldsymbol{E})$ is discussed in [17–21]. $\boldsymbol{E} \cdot (\nabla \times \boldsymbol{E})$ is referred to in [19] as the local helicity density of the vector field $\boldsymbol{E}$, and 'helicity' is often used [22–27] in this context. We prefer *chiral* density and *chiral* current, since helicity in particle physics is the projection of the angular momentum onto the momentum direction. We use the particle physics definition here, in particular as applied to photons in (1.3) below.

Thus, in the terminology of this paper, χ is the sum of the electric and magnetic field chiral densities. It is clear that the chiral density is maximum when the fields are eigenvectors of curl, with the same eigenvalue, $\nabla \times \boldsymbol{E} = k\boldsymbol{E}$, $\nabla \times \boldsymbol{B} = k\boldsymbol{B}$. Self-dual fields are eigenvectors of curl, as we shall see in section 3. To anticipate results to be derived, it follows from (1.1) that *the chiral density of self-dual fields is maximal, and proportional to the energy density*. Likewise it follows from (1.2) that *the chiral current of self-dual fields is maximal, and proportional to the momentum density* (or the Poynting vector).

Bliokh ad Nori [13] have shown by means of a Fourier representation of the electric and magnetic fields that the ratio of the chiral density to the energy density in monochromatic beams of angular frequency $\omega = ck$ lies between $\pm 8\pi k$ (using our definition of chiral density): see their equation (18), repeated as (4.6) here. We show in section 4 that for self-dual fields the ratio

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has to be exactly $\pm 8\pi k$. This gives an alternative proof of the chiral density being maximal for self-dual fields.

Coles and Andrews [14, 15] also use a Fourier decomposition (of the vector potential) and obtain an expression for the total chiral content of a given electromagnetic field as a sum over the difference between the numbers of left and right circularly polarized photons in the field (equivalently: the difference between numbers of photons of opposite helicity). This expression, equation (13) of [14] or equation (9) of [15] is

$$\int d^3r\, \chi = \hbar c \sum_{\boldsymbol{k}} k^2 \{N_L(\boldsymbol{k}) - N_R(\boldsymbol{k})\}. \tag{1.3}$$

A molecular analog of (1.3) is the statement that the rotatory power of a solution is proportional to the difference between the numbers of molecules of opposite handedness in the solution. Equation (1.3) can be applied in principle to the calculation of the total chiral content of a given electromagnetic pulse, calculated explicitly in [36]. However, electromagnetic beams have infinite extent (at least theoretically), and what we need for beams is the chiral content *per unit length of beam*, which is a finite quantity, and related here to the energy content per unit length of beam U' defined by (2.2), and given in (2.3) for TE and TM beams, for example. We note that the Fourier decomposition of fields is completely general, and quantities like the chiral content involve summation or integration over *three* wavevector components. In contrast, the expressions given here apply to directed beams and involve integration over *one* wavevector component only (either the longitudinal or the transverse component; these are linked since the total wavenumber $k = \omega/c$ is fixed for a given frequency of the monochromatic beams being considered).

Barnett *et al* [22] define an 'optical helicity' which in general is distinct from the chiral density defined here, but proportional to it for monochromatic fields. Their 'helicity' (different from the particle physics helicity) is defined in terms of the vector potential $\boldsymbol{A}$, the curl of which gives $\boldsymbol{B}$, and another vector potential the curl of which gives $-\boldsymbol{E}$. The same definition of helicity is adopted by Bliokh *et al* [27]. The title of their paper is 'dual electromagnetism: helicity, spin, momentum and angular momentum'. The title of [16] by Cameron *et al* is 'chirality and the angular momentum of light', but the authors note that 'chirality is the concept of handedness while the angular momentum of light, in particular spin, is associated with rotation rather than any form of inversion'. In fact it follows from the results of [36] that there is no *universal* relation between chirality and angular momentum: for example, general expressions applicable to all causal TE and TM pulses, backed up by explicit calculation for particular waveforms, show that all such pulses have zero total chiral content, whether or not they carry angular momentum. What equations like (1.3) show is that chiral content is related to the difference between numbers of photons of opposite *helicity*, defined as the projection of the angular momentum onto the momentum direction of the photon.

In the following, we summarize the properties of electromagnetic beams in sections 2 and 3, and note the special properties of self-dual beams. Section 4 deals with chiral measures of electromagnetic waves, and their special properties in monochromatic self-dual beams. Section 5 discusses three families of beams as examples.

2. Causal electromagnetic beams

Recent work has explored causal solutions of the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$, which may be written as superpositions of Bessel beams [28–30]. By *causal* we mean without backward propagation far from the focal region. (There is in general backflow associated with the zeros of the beam wavefunction; these zeros lie in the focal region.) The general expression for monochromatic beams of frequency $\omega = ck$ contains the wavenumber weight function $f(k, \kappa)$:

$$\psi(\boldsymbol{r}, k) = e^{im\phi} \int_0^k d\kappa\, f(k, \kappa) e^{iqz} J_m(\kappa\rho),\; q = \sqrt{k^2 - \kappa^2}. \tag{2.1}$$

We are using cylindrical polar coordinates (ρ, ϕ, z), with $\rho = (x^2 + y^2)^{\frac{1}{2}}$ the distance from the z-axis, and ϕ the azimuthal angle. The transverse and longitudinal wavenumber components $\kappa = k_\rho$ and $q = k_z$ are constrained by $\kappa^2 + q^2 = k^2$, and $J_m(\kappa\rho)$ is the regular Bessel function of order m.

The function $f(k, \kappa)$, in general complex, is subject only to the existence of (2.1) and associated integrals, for example those which give the energy, momentum and angular momentum contained in a transverse slice of a beam constructed from $\psi(\rho, \phi, z)$. The form of (2.1) guarantees the absence of asymptotic backward propagation: the integrand contains the factor $e^{i(qz-kct)}$, with $k \geqslant q \geqslant 0$. For ψ to be dimensionless, the dimension of $f(k, \kappa)$ is to be that of a length.

Free-space electromagnetic beams may be constructed from solutions of the Helmholtz equation. In [28] TE and TM beams were considered, together with the self-dual TM + iTE beams, and their total energy, momentum and angular momentum per unit length of the beam were found as integrals over the weight function $f(k, \kappa)$. For beams propagating in the z direction (but spreading or converging transversely, as all transversely localized beams do) the main conserved quantities are the energy, momentum and angular momentum *per unit length* of the beam. We denote by U' the energy per unit length, and likewise for P_z', J_z'. (The component of interest is J_z, since it is intrinsic to the beam, unchanged by a shift of origin.) Thus, for example, $U'dz$ is the energy content in a slice of thickness dz of the beam:

$$U' = \int d^2r\, u = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, u = \int_0^{\infty} d\rho\, \rho \int_0^{2\pi} d\phi\, u \tag{2.2}$$

(u is the energy density, defined below in (3.5)). Reference [28] shows that, for TE and TM beams

$$\begin{bmatrix} U' \\ cP_z' \\ cJ_z' \end{bmatrix} = \frac{E_0^2}{4k^3} \int_0^k d\kappa\, |f(k, \kappa)|^2 \kappa \begin{bmatrix} k \\ q \\ m \end{bmatrix} \quad \text{(TE or TM)}. \tag{2.3}$$

(A field amplitude factor $k^{-1}E_0$ has been inserted to harmonize the notation of [29] and the present paper.) The results given in (2.3) are based entirely on classical electrodynamics, but show that an electromagnetic TE or TM beams can be viewed as a superposition of photons with energies $\hbar ck$, z component of momentum $\hbar k_z = \hbar q$, and z component of angular momentum $\hbar m$, where m is the azimuthal winding number, as defined in (2.1).

The corresponding results for self-dual TM + iTE beams are twice those in (2.3) for the energy and momentum, but with an extra term in the angular momentum integral:

$$\begin{bmatrix} U' \\ cP_z' \\ cJ_z' \end{bmatrix} = \frac{E_0^2}{2k^3}\int_0^k d\kappa\, |f(k,\kappa)|^2\kappa \begin{bmatrix} k \\ q \\ m + \dfrac{\kappa^2}{2kq} \end{bmatrix} \quad (\mathrm{TM} + i\mathrm{TE}). \tag{2.4}$$

A similar result was derived in [29] for self-dual 'circularly polarized' beams, namely those that, in the plane-wave limit, have transverse electric and magnetic field components of equal amplitude and in phase quadrature. When the field amplitude is taken to be E_0, and for the self-dual 'CP' beam defined in [29], the corresponding result is

$$\begin{bmatrix} U' \\ cP_z' \\ cJ_z' \end{bmatrix} = \frac{E_0^2}{8k^3}\int_0^k d\kappa\, |f(k,\kappa)|^2 (k+q)^2\kappa^{-1} \begin{bmatrix} k \\ q \\ m + 1 + \dfrac{\kappa^2}{2kq} \end{bmatrix} \quad (\text{'CP'}). \tag{2.5}$$

Thus the 'CP' beam can also be viewed as a superposition of photons with energies $\hbar ck$ and z component of momentum $\hbar q$. However, as in the TM + iTE case, the z component of angular momentum is no longer simply $\hbar m$.

The results (2.4) and (2.5) rest on some intricate manipulation of highly singular integrals over products of Bessel functions. This analysis has been checked by the author against known beam wavefunctions with $m = 0, 1$ based on the 'proto-beam' [29], discussed also in [30]. The proto-beam has recently been shown to be the most tightly focused of all possible beams, according to an intensity criterion [31]. The $m = 1$ beam is obtained from the proto-beam by differentiation with respect to ρ. For both cases we find exact agreement with the results given in (2.4) and (2.5).

3. Monochromatic electromagnetic beams, self-duality

Electric and magnetic fields can be expressed in terms of the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ and scalar potential $V(\boldsymbol{r}, t)$ via

$$\boldsymbol{E} = -\nabla V - \partial_{ct}\boldsymbol{A}, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}. \tag{3.1}$$

With these substitutions the source-free Maxwell equations $\nabla\cdot\boldsymbol{B} = 0$, $\nabla\times\boldsymbol{E} + \partial_{ct}\boldsymbol{B} = 0$ are satisfied automatically. If further $\boldsymbol{A}$ and V satisfy the Lorenz condition $\nabla\cdot\boldsymbol{A} + \partial_{ct}V = 0$, substitution of (3.1) into Maxwell's free space equations (of which the curl equations couple $\boldsymbol{E}$ and $\boldsymbol{B}$), decouples the vector and the scalar potentials:

$$\nabla^2\boldsymbol{A} - \partial_{ct}^2\boldsymbol{A} = 0, \quad \nabla^2 V - \partial_{ct}^2 V = 0. \tag{3.2}$$

We now specialize to monochromatic beams, with time-dependence $e^{-i\omega t}$ everywhere. Real electric and magnetic fields are obtained by taking real or imaginary parts of the complex amplitudes times $e^{-i\omega t}$; for example taking the real part of the complex amplitude $\boldsymbol{E}(\boldsymbol{r})$ times $e^{-i\omega t}$ gives

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}, t) &= Re\{\boldsymbol{E}(\boldsymbol{r})e^{-i\omega t}\} = Re\{(\boldsymbol{E}_r + i\boldsymbol{E}_i)e^{-i\omega t}\} \\ &= \boldsymbol{E}_r\cos\omega t + \boldsymbol{E}_i\sin\omega t. \end{aligned} \tag{3.3}$$

We shall write the angular frequency as $\omega = ck$. Then from (3.2) the complex scalar $V(\boldsymbol{r})$ and all components of the complex vector $\boldsymbol{A}(\boldsymbol{r})$ satisfy the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$ (satisfied by the causal wavefunctions (2.1)). The Lorenz condition reads $\nabla\cdot\boldsymbol{A} - ikV = 0$. Hence the monochromatic beam complex electric and magnetic amplitudes $\boldsymbol{E}(\boldsymbol{r}), \boldsymbol{B}(\boldsymbol{r})$ can be obtained from the vector potential only:

$$\boldsymbol{E} = i[k\boldsymbol{A} + k^{-1}\nabla(\nabla\cdot\boldsymbol{A})], \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}. \tag{3.4}$$

The energy, momentum and angular momentum densities are, for real fields $\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{B}(\boldsymbol{r}, t)$ and in Gaussian units

$$\begin{aligned} u(\boldsymbol{r}, t) &= \frac{1}{8\pi}(E^2 + B^2), \\ \boldsymbol{p}(\boldsymbol{r}, t) &= \frac{1}{4\pi c}\boldsymbol{E}\times\boldsymbol{B}, \qquad \boldsymbol{j}(\boldsymbol{r}, t) = \boldsymbol{r}\times\boldsymbol{p}. \end{aligned} \tag{3.5}$$

The energy density in general oscillates at 2ω: from (3.3) $\boldsymbol{E}(\boldsymbol{r}, t)^2 = E_r^2\cos^2\omega t + 2\boldsymbol{E}_r.\boldsymbol{E}_i\cos\omega t\sin\omega t + E_i^2\sin^2\omega t$, which cycle-averages to $\frac{1}{2}(E_r^2 + E_i^2)$. Likewise the energy density derived from $Im\{\boldsymbol{E}(\boldsymbol{r})e^{-i\omega t}\}$ averages to the same. For the momentum density we find the time average of $\boldsymbol{E}\times\boldsymbol{B}$ to be $\frac{1}{2}(\boldsymbol{E}_r\times\boldsymbol{B}_r + \boldsymbol{E}_i\times\boldsymbol{B}_i)$, when either the real or the imaginary parts are used. Hence whether we take the real or imaginary parts of the complex fields, the cycle-averaged energy and momentum densities in free space, in terms of the complex space-dependent amplitudes $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_r + i\boldsymbol{E}_i$, $\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}_r + i\boldsymbol{B}_i$, are given by

$$\begin{aligned} \bar{u}(\boldsymbol{r}) &= \frac{1}{16\pi}(E_r^2 + E_i^2 + B_r^2 + B_i^2) \\ &= \frac{1}{16\pi}(\boldsymbol{E}\cdot\boldsymbol{E}^* + \boldsymbol{B}\cdot B^*) \\ c\bar{\boldsymbol{p}}(\boldsymbol{r}) &= \frac{1}{8\pi}(\boldsymbol{E}_r\times\boldsymbol{B}_r + \boldsymbol{E}_i\times\boldsymbol{B}_i) \\ &= \frac{1}{16\pi}(\boldsymbol{E}\times\boldsymbol{B}^* + \boldsymbol{E}^*\times\boldsymbol{B}). \end{aligned} \tag{3.6}$$

The *duality* transformation $\boldsymbol{E}\to\boldsymbol{B}$, $\boldsymbol{B}\to-\boldsymbol{E}$ leaves the energy and momentum densities unchanged. The more general transformation is $\boldsymbol{E}\to\boldsymbol{E}\cos\theta + \boldsymbol{B}\sin\theta$, $\boldsymbol{B}\to\boldsymbol{B}\cos\theta - \boldsymbol{E}\sin\theta$; here we shall use only the values $\theta = \pm\pi/2$. It is known [32, 33] that invariance of Maxwell's equations under the general duality transformation leads to a conservation law, namely that of the difference between right and left polarized photons in a given electromagnetic field; [33] relates this to the chiral measures of the next section.

Self-dual electromagnetic fields are unchanged by the duality transformation. Complex field amplitudes which satisfy $\boldsymbol{E} = \pm i\boldsymbol{B}$ are self-dual. They have particularly simple properties because of the symmetry between the electric and magnetic fields. In self-dual monochromatic (or 'steady') beams the electromagnetic energy and momentum densities do not oscillate in time ([3], section 4), whereas as we saw above they normally oscillate at twice the angular frequency of the beam. When $\boldsymbol{E} = \pm i\boldsymbol{B}$ we have

$$\boldsymbol{E}_r + i\boldsymbol{E}_i = \pm i(\boldsymbol{B}_r + i\boldsymbol{B}_i), \ \boldsymbol{E}_r = \mp\boldsymbol{B}_i, \ \boldsymbol{E}_i = \pm\boldsymbol{B}_r. \tag{3.7}$$

The energy and momentum densities of self-dual monochromatic beams are thus time-independent, and simplify to

$$\begin{aligned} u(\boldsymbol{r}) &= \frac{1}{8\pi}\boldsymbol{E}\cdot\boldsymbol{E}^* = \frac{1}{8\pi}\boldsymbol{B}\cdot\boldsymbol{B}^* \\ &= \frac{1}{8\pi}(E_r^2 + E_i^2) = \frac{1}{8\pi}(B_r^2 + B_i^2), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \boldsymbol{p}(\boldsymbol{r}) &= \pm\frac{i}{8\pi c}\boldsymbol{E}\times\boldsymbol{E}^* = \pm\frac{i}{8\pi c}\boldsymbol{B}\times\boldsymbol{B}^* \\ &= \pm\frac{1}{8\pi c}\boldsymbol{E}_r\times\boldsymbol{E}_i = \pm\frac{1}{8\pi c}\boldsymbol{B}_r\times\boldsymbol{B}_i. \end{aligned} \tag{3.9}$$

We note finally that in monochromatic self-dual beams both the electric and magnetic amplitudes are *eigenvectors of curl* in free space [3]: $\nabla\times\boldsymbol{E} + \partial_{ct}\boldsymbol{B} = 0$ becomes $\nabla\times\boldsymbol{E} = ik\boldsymbol{B}$ $(k = \omega/c)$ because the time dependence is in the factor $e^{-i\omega t}$. Then when $\boldsymbol{E} = \pm i\boldsymbol{B}$ for complex amplitudes, for the real fields $\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{B}(\boldsymbol{r}, t)$ we have

$$\nabla\times\boldsymbol{E} = \pm k\boldsymbol{E}, \ \nabla\times\boldsymbol{B} = \pm k\boldsymbol{B}. \tag{3.10}$$

The real and imaginary parts of the field amplitudes are separately eigenvectors of the curl operator: for monochromatic fields the Maxwell equation $\nabla\times\boldsymbol{E} + \partial_{ct}\boldsymbol{B} = 0$ reads

$$\begin{aligned} &\nabla\times(\boldsymbol{E}_r\cos\omega t + \boldsymbol{E}_i\sin\omega t) \\ &\quad + \partial_{ct}(\boldsymbol{B}_r\cos\omega t + \boldsymbol{B}_i\sin\omega t) = 0. \end{aligned} \tag{3.11}$$

For self-dual fields substitution of $\boldsymbol{B}_r = \pm\boldsymbol{E}_i$, $\boldsymbol{B}_i = \mp\boldsymbol{E}_r$ gives us, on equating coefficients of $\cos\omega t$, $\sin\omega t$,

$$\nabla\times\boldsymbol{E}_r = \pm k\boldsymbol{E}_r, \ \nabla\times\boldsymbol{E}_i = \pm k\boldsymbol{E}_i. \tag{3.12}$$

Hence real and imaginary parts of the electric field complex amplitude are eigenvectors of curl. Likewise the real and imaginary parts of the magnetic field amplitude are eigenvectors of curl, with the same eigenvalue. (Various aspects of eigenstates of curl are discussed in [17–21].) The fact that for self-dual beams the fields are eigenvectors of curl has consequence for chirality, as we shall see next.

4. Chirality

Lipkin's 'zilch' [4–10] has been suggested as a measure of the chirality of light, important in the interaction of light with chiral matter [11–16]. There are two quantities, a chiral density $\chi(\boldsymbol{r}, t)$ and a chiral current $\boldsymbol{C}(\boldsymbol{r}, t)$. For real fields $\boldsymbol{E}(\boldsymbol{r}, t)$, $\boldsymbol{B}(\boldsymbol{r}, t)$ these are defined as

$$\begin{aligned} \chi &= \boldsymbol{E}\cdot(\nabla\times\boldsymbol{E}) + \boldsymbol{B}\cdot(\nabla\times\boldsymbol{B}), \\ \boldsymbol{C} &= \boldsymbol{E}\times(\nabla\times\boldsymbol{B}) - \boldsymbol{B}\times(\nabla\times\boldsymbol{E}). \end{aligned} \tag{4.1}$$

The free-space Maxwell equations imply that $\chi(\boldsymbol{r}, t)$ and $\boldsymbol{C}(\boldsymbol{r}, t)$ satisfy the conservation law

$$\partial_{ct}\chi + \nabla\cdot\boldsymbol{C} = 0. \tag{4.2}$$

We can replace the curls in (4.1) by time derivatives, from Maxwell's free-space curl equations $\nabla\times\boldsymbol{E} + \partial_{ct}\boldsymbol{B} = 0$, $\nabla\times\boldsymbol{B} - \partial_{ct}\boldsymbol{E} = 0$. For *monochromatic* fields with time dependence as in (3.3) the time derivatives will bring down a factor of $k = \omega/c$. The consequence is that χ and $\boldsymbol{C}$ are independent of time, as noted in [4, 13]. This is in contrast to the energy and momentum densities, which in general oscillate at twice the angular frequency ω of the fields.

From (4.2) we see that the fact that χ is independent of time in monochromatic fields implies that the divergence of $\boldsymbol{C}$ is zero. It follows that the integral of the chiral current component along the beam direction through a section of a monochromatic electromagnetic beam is constant along the beam. The proof is the same as for the other invariant quantities which follow from conservation laws [28, 34]. Since the chiral density and current in monochromatic beams are independent of time, there is no need for a cycle average of (4.2).

For *self-dual* beams the chiral density and current are strictly proportional to the energy and momentum densities, respectively: from (3.10) to (4.1) we have

$$\begin{aligned} \chi &= \pm k(E^2 + B^2) = \pm(8\pi k)u, \\ \boldsymbol{C} &= 2k\ \boldsymbol{E}\times\boldsymbol{B} = (8\pi k)c\boldsymbol{p}. \end{aligned} \tag{4.3}$$

Another route to (4.3) is via the real and imaginary parts of the complex field amplitudes. Self-dual beams have complex amplitudes related by $\boldsymbol{E}(\boldsymbol{r}) = \pm i\boldsymbol{B}(\boldsymbol{r})$ or $\boldsymbol{E}_r + i\boldsymbol{E}_i = \pm i(\boldsymbol{B}_r + i\boldsymbol{B}_i)$ and so $\boldsymbol{B}_r = \pm\boldsymbol{E}_i$, $\boldsymbol{B}_i = \mp\boldsymbol{E}_r$. In terms of the real and imaginary parts of the complex amplitudes

$$\begin{aligned} \chi(\boldsymbol{r}) &= k(\boldsymbol{B_r}\cdot\boldsymbol{E_i} - \boldsymbol{E_r}\cdot\boldsymbol{B_i}), \\ \boldsymbol{C}(\boldsymbol{r}) &= k(\boldsymbol{E_r}\times\boldsymbol{E_i} + \boldsymbol{B_r}\times\boldsymbol{B_i}). \end{aligned} \tag{4.4}$$

For self-dual field amplitudes with $\boldsymbol{E}(\boldsymbol{r}) = \pm i\boldsymbol{B}(\boldsymbol{r})$ there is a further reduction of (4.4) to

$$\begin{aligned} \chi &= \pm k(E_r^2 + E_i^2) = \pm k(B_r^2 + B_i^2), \\ \boldsymbol{C} &= 2k(\boldsymbol{E_r}\times\boldsymbol{E_i}) = 2k(\boldsymbol{B_r}\times\boldsymbol{B_i}). \end{aligned} \tag{4.5}$$

Comparison with the self-dual densities in (3.8) and (3.9) verifies (4.3).

Hence *self-dual monochromatic fields have chiral density and chiral current strictly proportional to the energy density and c times the momentum density*, with the same proportionality constant ($8\pi k$ for the Gaussian units used here).

Note that the sign of the chiral density is that in the complex amplitude relation $\boldsymbol{E}(\boldsymbol{r}) = \pm i\boldsymbol{B}(\boldsymbol{r})$, while the chiral current is directly proportional to $\boldsymbol{p}$.

The similarity between χ and u and between $\boldsymbol{C}$ and $\boldsymbol{p}$ has been noted before [13]. What we have shown is that for self-dual monochromatic beams the relationship is strict proportionality, for both pairs of quantities. In electromagnetic pulses the $8\pi k$ factor also converts energy content to chiral content, but occurs inside the wavenumber integral (appendix B of chapter 3 in [35], [36]).

It is clear from (4.1) that χ will be maximum for given field strengths when the curls are parallel to the respective fields, or in other words, when the fields are eigenstates of the curl operator as in (3.10). It follows from the definitions of the chiral densities in (4.1) that *self-duality* of monochromatic fields implies *maximal chirality*, in the following sense: at any point in space the fact that the fields are eigenvectors of curl maximizes the local chiral density. The *global* maximum of both the energy and the chiral densities will be in the focal region. Different types of electromagnetic beams will have different forms of energy density, and self-dual beams will have chiral density everywhere proportional to the energy density.

The maximality of the magnitude the chiral density also follows from the work of Bliokh and Nori [13], who have shown by means of a Fourier representation of the electric and magnetic fields that the ratio of the chiral density to the energy density lies between $\pm 8\pi k$. Their equation (18) reads (using our definition of chiral density)

$$\frac{\chi}{8\pi\overline{u}} = \frac{\overline{u_+} - \overline{u_-}}{\overline{u_+} + \overline{u_-}}, \text{ so } -1 \leqslant \frac{\chi}{8\pi\overline{u}} \leqslant 1. \tag{4.6}$$

The symbols $\overline{u_+}$, $\overline{u_-}$ denote energy densities for fields with positive and negative helicities, respectively, cycle-averaged. We have shown above that for self-dual fields the ratio has to be exactly $\pm 8\pi k$. Thus, we have a second proof of the theorem that *the chiral density is maximal for self-dual fields.*

5. Examples

As the simplest example of (null) chirality, consider the TM monochromatic beam, with vector amplitude along the beam axis, $A_{\mathrm{TM}} = [0, 0, \psi] = (0, 0, \psi)$. We shall give expressions in both Cartesian $[x, y, z]$ and cylindrical (ρ, ϕ, z) coordinates; in general the ϕ dependence of ψ is in the factor $e^{im\phi}$, as in (2.1). We shall assume at first that the beam wavefunction does not depend on the azimuthal angle, and set $\psi = \psi(\rho, z) = \psi_r(\rho, z) + i\psi_i(\rho, z)$, $m = 0$ in (2.1). Then the complex magnetic amplitude is $\boldsymbol{B} = \nabla \times \boldsymbol{A} = (0, -\partial_\rho\psi, 0)$, and the magnetic field is everywhere azimuthal and is linearly polarized, since the real and imaginary parts are collinear. The magnetic field lines are circles, with centers on the beam axis. The complex electric amplitude has radial and longitudinal components, $\boldsymbol{E} = ik^{-1}(\partial_\rho\partial_z, 0, \partial_z^2 + k^2)\psi$, and the electric field is elliptically polarized in general. The electric and magnetic fields are everywhere perpendicular. The azimuthal component of the momentum density is zero, and therefore so is the angular momentum. The chiral density χ is zero, being proportional to the winding number m. This statement is true also for electromagnetic TE and TM pulses [36].

The chiral current of $m = 0$ TM beams is everywhere azimuthal, with C_ϕ proportional to

$$[(\partial_z^2 + k^2)\psi_r](\partial_\rho\partial_z\psi_i) - [(\partial_z^2 + k^2)\psi_i](\partial_\rho\partial_z\psi_r). \tag{5.1}$$

In the plane-wave limit, $\psi \to e^{ikz}$, the one non-zero chiral component C_ϕ vanishes. In fact the whole beam vanishes in this limit.

In contrast, the self-dual TM + iTE beams are based on the vector amplitude [2, 28]

$$\begin{aligned}\boldsymbol{A} = \boldsymbol{A}_{\mathrm{TM}} + i\boldsymbol{A}_{\mathrm{TE}} &= k^{-1}[\partial_y, -\partial_x, k]\psi \\ &= k^{-1}(\rho^{-1}\partial_\phi, -\partial_\rho, k)\psi.\end{aligned} \tag{5.2}$$

With $m = 0$ as in the TM beam example, the complex field amplitudes are

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = k^{-1}(\partial_\rho\partial_z, -k\partial_\rho, \partial_z^2 + k^2)\psi, \; \boldsymbol{E} = i\boldsymbol{B} \tag{5.3}$$

(TM − iTE beams have $\boldsymbol{E} = -i\boldsymbol{B}$). Because this beam is self-dual, the results (4.5) give the maximal (for this kind of beam) chirality measures χ and $\boldsymbol{C}$, respectively proportional to the energy and momentum densities. The chiral density and current are not zero even when there is no azimuthal dependence in the scalar beam wavefunction.

The TM, TE and TM $\pm$ iTE beams all vanish in the plane wave limit. Out final beam family does not. These are the 'circularly polarized' beams, which we expect to have the largest angular momenta and the largest chiral content, for given energy content. (We do not have proof for these intuitive expectations.) There is also an intrinsic interest to this family of beams, given that the textbook circularly polarized electromagnetic beam does not exist, except as a theoretical limit: a beam of infinite breadth. To put it more precisely, a beam which is everywhere circularly polarized in the same plane cannot be transversely localized ([37] section 2, [38] section 20.1.3). As soon as we constrain light transversely to its net propagation, interesting structures appear. Most prominent among these are the phase singularities in the focal region [39–42]. There are also consequences of transverse localization on the degree of polarization, and these were explored in [29], with emphasis on the focal region. It is a remarkable fact that a transversely localized beam which would have perfect circular polarization in the plane-wave limit, has rings of perfect *linear* polarization surrounding the circular polarization region in the focal plane.

Although 'circularly polarized' beams are (in vector potential and electric and magnetic fields) superpositions of beams polarized linearly along perpendicular transverse axes and in phase quadrature, the resultant energy and momentum densities, chirality measures and polarization measure are *not* superpositions, because all of these quantities are quadratic or bilinear in the fields.

The vector potential corresponding to that of a self-dual 'circularly polarized' beam is given in [29, 37], and in equation (5.6) below. We shall give expressions in both Cartesian $[x, y, z]$ and polar (ρ, ϕ, z) coordinates; it is assumed that the ϕ dependence of ψ is in the factor $e^{im\phi}$, as in (2.1). In that case we have

$$\begin{aligned}\partial_x &= \cos\phi\,\partial_\rho - \rho^{-1}\sin\phi\,\partial_\phi \rightarrow \cos\phi\,\partial_\rho - im\rho^{-1}\sin\phi\\ \partial_y &= \sin\phi\,\partial_\rho + \rho^{-1}\cos\phi\,\partial_\phi \rightarrow \sin\phi\,\partial_\rho + im\rho^{-1}\cos\phi\\ \partial_x + i\partial_y &= e^{i\phi}(\partial_\rho + i\rho^{-1}\partial_\phi) \rightarrow e^{i\phi}(\partial_\rho - m\rho^{-1}).\end{aligned} \tag{5.4}$$

Consider first the complex vector amplitude $\boldsymbol{A}_1 = k^{-1}E_0[-i, 1, 0]\psi = k^{-1}E_0 e^{i\phi}(-i, 1, 0)\psi$. Note the additional factor $e^{i\phi}$ which appears in the polar coordinate form. The field amplitudes derived from this vector potential are

$$\begin{aligned}\boldsymbol{B}_1 &= k^{-1}E_0[-\partial_z, -i\partial_z, \partial_x + i\partial_y]\psi\\ &= k^{-1}E_0 e^{i\phi}(-\partial_z, -i\partial_z, \partial_\rho - m\rho^{-1})\psi\\ \boldsymbol{E}_1 &= k^{-2}E_0[\partial_x(\partial_x + i\partial_y) + k^2, \partial_y(\partial_x + i\partial_y)\\ &\quad + ik^2, (\partial_x + i\partial_y)\partial_z]\psi\\ &= k^{-2}E_0 e^{i\phi}(\partial_\rho^2 - m\rho^{-1}\partial_\rho + k^2 + m\rho^{-2},\\ &\quad i[(m+1)\rho^{-1}\partial_\rho + k^2 - m(m+1)\rho^{-2}],\\ &\quad \partial_\rho\partial_z - m\rho^{-1}\partial_z)\psi.\end{aligned} \tag{5.5}$$

In the plane wave limit $\psi \rightarrow e^{ikz}$, $\boldsymbol{B}_1 \rightarrow E_0[-i, 1, 0]e^{ikz}$, $\boldsymbol{E}_1 \rightarrow E_0[1, i, 0]e^{ikz}$. This is the textbook circularly polarized plane wave (of positive helicity), in which the electric and magnetic fields and the propagation direction are mutually perpendicular, and the electric and magnetic fields are in phase quadrature. Theorem 2.3 of [37] shows that this only possible in the plane wave limit: transversely finite beams which are everywhere circularly polarized in a fixed plane do not exist.

The beam derived from the vector potential $\boldsymbol{A}_1$ has $\boldsymbol{B}_1$ with different polarization properties to that of $\boldsymbol{E}_1$. Also the electromagnetic energy and momentum densities defined in (3.5) oscillate in time, at angular frequency $2\omega = 2ck$. Self-dual beams (section 3), in which the complex field amplitudes satisfy $\boldsymbol{E} = \pm i\boldsymbol{B}$ and the energy and momentum densities do not oscillate in time, may be constructed from the above complex vector amplitude. We take

$$\begin{aligned}\boldsymbol{A} &= \frac{1}{2}(\boldsymbol{A}_1 + k^{-1}\nabla\times\boldsymbol{A}_1)\\ &= \frac{1}{2}k^{-2}E_0[-(\partial_z + ik), -i(\partial_z + ik), \partial_x + i\partial_y]\psi\\ &= \frac{1}{2}k^{-2}E_0 e^{i\phi}(-(\partial_z + ik), -i(\partial_z + ik), \partial_\rho - m\rho^{-1})\psi.\end{aligned} \tag{5.6}$$

The corresponding complex magnetic amplitude $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ is

$$\begin{aligned}\boldsymbol{B} &= \frac{1}{2}k^{-2}E_0[\partial_y(\partial_x + i\partial_y) + i(\partial_z + ik)\partial_z,\\ &\quad -\partial_x(\partial_x + i\partial_y) - (\partial_z + ik)\partial_z,\\ &\quad -i(\partial_x + i\partial_y)(\partial_z + ik)]\psi\\ &= \frac{1}{2}k^{-2}E_0 e^{i\phi}(i(m+1)\rho^{-1}(\partial_\rho - m\rho^{-1}) + i(\partial_z + ik)\partial_z,\\ &\quad -(\partial_\rho^2 - m\rho^{-1}\partial_\rho + m\rho^{-2}) - (\partial_z + ik)\partial_z,\\ &\quad -i(\partial_\rho - m\rho^{-1})(\partial_z + ik))\psi.\end{aligned} \tag{5.7}$$

The complex electric amplitude is $\boldsymbol{E} = i\boldsymbol{B}$, for waves satisfying the Helmholtz equation. This may be verified from the free-space, time-harmonic version of Ampere's law, $\boldsymbol{E} = \frac{i}{k}\nabla\times\boldsymbol{B}$. In the plane wave limit we regain the textbook circularly polarized plane wave, as we had before:

$$\psi \rightarrow e^{ikz}, \boldsymbol{B} \rightarrow E_0[-i, 1, 0]e^{ikz}, \ \boldsymbol{E} \rightarrow E_0[1, i, 0]e^{ikz}. \tag{5.8}$$

The real parts of the limiting amplitudes given in (5.8) give $\boldsymbol{E}\times\boldsymbol{B} = E_0^2[0, 0, 1]$, and $j_z = (4\pi c)^{-1}[\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B})]_z = 0$. A finite z component of the angular momentum can result from a transversely finite beam. When the beam is nearly a plane wave but transversely localized one finds $ckJ_z' \approx U'$ ([43], problems 7.20 and 7.21). Hence we can associate a *positive helicity* with beams that approach the limit (5.8) within a transversely localized region.

6. Summary

We have shown that the chiral density and the chiral current in self-dual monochromatic beams are proportional to the energy and momentum densities, respectively. The proportionality constants are $\pm 8\pi k$ and $8\pi kc$, where $k = \omega/c$. It follows that the chiral contents per unit length of beam are respectively proportional to the energy and momentum contents:

$$X' = \pm 8\pi k U', \quad C_z' = \int d^2r\, C_z = 8\pi kc P_z'. \tag{6.1}$$

The electric and magnetic fields in self-dual monochromatic beam are eigenvectors of curl, ensuring that the chiral density and current are maximal. Thus such beams are, theoretically at least, the best candidates for studying interactions of light with chiral matter.

Acknowledgments

The author is grateful for reviewer comments and suggestions.

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