**Brewster angles in reflection by uniaxial crystals**

John Lekner

Department of Physics, Victoria University of Wellington, Wellington, New Zealand

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Reflection by anisotropic media is characterized by the four reflection amplitudes $r_{pp}$, $r_{rs}$, $r_{sp}$, and $r_{ss}$. We show that $r_{ss}$ can be zero at angles $\theta_{pp}$, the anisotropic Brewster angles, and that a quantity related to $\theta_{pp}$ satisfies a quartic equation. When the refractive index of the medium of incidence lies between the ordinary and the extraordinary indices of the crystal, it is possible for $r_{ss}$ to be zero at an angle $\theta_{ss}$, and there exist four equivalent orientations of the crystal optic axis for which $r_{pp}$, $r_{rs}$, and either $r_{sp}$ or $r_{ss}$ are simultaneously zero, at angle of incidence equal to arctan $(n_3/n_1)$.

1. **INTRODUCTION**

It is well known\(^1\) that a p-polarized wave has zero reflection from an isotropic material at the Brewster angle $\theta_B$ given by $\tan \theta_B = n/n_1$, where $n$ is the refractive index of the (nonabsorbing) material and $n_1$ is the refractive index of the medium of incidence. Special cases of the analogous zero reflection of $p$-polarized incident waves into the $p$ polarization by uniaxial crystals are known\(^2\); here we will examine the general case of arbitrary orientation of the optic axis relative to the reflecting plane and to the plane of incidence. We find that it is possible to find the angles $\theta_{pp}$, at which the $r_{ss}$ reflection amplitude is zero, by solving a quartic equation. Surprisingly, the $r_{ss}$ reflection amplitude also can be zero when the refractive index of the medium of incidence lies between the ordinary and the extraordinary indices. The angle $\theta_{ss}$ at which this occurs is found by solving a quartic equation, which is related to the quartic equation that determines $\theta_{pp}$. Further, for particular orientation of the crystal optic axis and at an angle of incidence whose tangent equals the ratio of the ordinary index to the index of the medium of incidence, $r_{pp}$, $r_{ss}$, and one of $r_{sp}$ or $r_{ss}$ can be zero together.

2. **REFLECTION BY A UNIAXIAL CRYSTAL**

I will use the results of a recent paper\(^5\) on the optical properties of uniaxial crystals. Let $\theta$ be the angle of incidence at which an incident plane wave strikes a planar surface of a uniaxial crystal. The $x$ axis is defined as the inward normal to the reflecting surface, which lies in the $x$–$y$ plane. The $y$ axis lies in the plane of incidence, the $y$ axis normal to it. Then the $x$ components of the incident, the ordinary, and the extraordinary wave vectors $k_1$, $k_2$, and $k_e$ are identical, with value

$$K = k_1 \sin \theta = n_1 \frac{\omega}{c} \sin \theta. \quad (1)$$

The $z$ components of the incident and the reflected waves are $\pm q_1$, where

$$q_1^2 = k_1^2 - K^2 = k_1^2 \cos^2 \theta. \quad (2)$$

The wave vector $k_e$ of the ordinary wave inside the crystal has magnitude

$$k_e = n_e \frac{\omega}{c}, \quad (3)$$

and normal component $q_e$ given by

$$q_e^2 = k_e^2 - K^2. \quad (4)$$

The extraordinary wave vector has components $(K, 0, q_e)$, where

$$q_e = \sqrt{q^2 - K^2 q_e^4}, \quad (5)$$

where $q = \bar{q} - \alpha \gamma K \Delta \varepsilon / \varepsilon_\gamma$, and $q_\gamma = q_e = n_x^2 - n_y^2$, the optic axis has direction cosines $\alpha, \beta$, and $\gamma$ relative to the $x, y$, and $z$ axes, respectively, and

$$\bar{q}^2 = \varepsilon_0 \varepsilon_\gamma \omega^2 / c^2 - K^2 (\varepsilon_0 - \beta^2 \Delta \varepsilon) / \varepsilon_\gamma, \quad (6)$$

with $\varepsilon_\gamma = n_\gamma^2$ defined as

$$\varepsilon_\gamma = \varepsilon_0 + \gamma^2 \Delta \varepsilon. \quad (7)$$

The reflection amplitudes found in Ref. 5 may be written in the form

$$r_{ss} = \frac{a(q_1 - q_e) + b(q_1 + q_e)}{a(q_1 + q_e) + b(q_1 + q_e)}, \quad (8)$$

$$-r_{pp} = \frac{a'(q_1 + q_e) + b'(q_1 + q_e)}{a(q_1 + q_e) + b(q_1 + q_e)}, \quad (9)$$

$$r_{sp,ps} = \frac{2b'(a q_e + \gamma K) (q_e - q_s) k_1 k_e^2}{a(q_1 + q_e) + b(q_1 + q_e)}, \quad (10)$$

(11)

where

$$a = (a q_e - \gamma K) [a(k_x^2 q_e + q_t q_e^2) - \gamma K (k_x^2 + q_t q_e)],$$

$$a' = (a q_e - \gamma K) [a(k_x^2 q_e + q_t q_e^2) - \gamma K (k_x^2 - q_t q_e)],$$

$$b = \beta^2 k_x^2 (k_x^2 + q_t q_e), \quad b' = \beta^2 k_x^2 (k_x^2 - q_t q_e),$$

and $q_t = k_1^2 / q_1$. We note that the reflection amplitudes are particularly simple when $\beta = 0$ (that is, when the optic axis lies in the plane of incidence); then all $a, b', r_{sp}$, and

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Reflection amplitudes reduce to

\[ r_\alpha (\beta = 0) = \frac{q_1 - q_o}{q_1 + q_o}, \quad r_{pp} (\beta = 0) = \frac{Q - Q_1}{Q + Q_1}, \]  

with \( Q_1 = q_1/e_1, \) \( Q = q_\gamma/n_1n_2, \) where

\[ q_\gamma^2 = \epsilon_o \omega^2/\epsilon^2 - K^2. \]  

In this case the Brewster angle (at which \( r_{pp} \) is zero) is found by solving \( Q = Q_1, \) which gives

\[ \tan^2 \theta_{pp} (\beta = 0) = \frac{e_o(e_\gamma - e_1)}{e_1(e_\gamma - e_1)}, \]  

This formula contains as special cases two known results for the Brewster angle when the optic axis coincides with the \( x \) and the \( z \) axes:

\[ \tan^2 \theta_{pp} (\alpha^2 = 1) = \frac{e_o(e_\gamma - e_1)}{e_1(e_\gamma - e_1)}, \]

\[ \tan^2 \theta_{pp} (\gamma^2 = 1) = \frac{e_o(e_\gamma - e_1)}{e_1(e_\gamma - e_1)}. \]

These two expressions give bounds on the Brewster angle.

3. Determination of \( \theta_{pp} \)

The Brewster angle \( \theta_{pp} \) is that angle at which the numerator of Eq. (9) is zero. This numerator contains the angle of incidence in the variable \( K, \) which appears linearly in \( q_\gamma \) and \( q_\alpha \) and also inside a square root in \( q_\gamma, q_1, \) and \( q_o. \) An algebraic equation can be obtained by use of a symmetry of \( r_{pp} \) and by introducing a variable related to \( K^2 \) through a bilinear transformation.

The reflection amplitudes \( r_{pp} \) and \( r_\alpha \) are unchanged by change of sign of any of the three optic direction cosines \( \alpha, \beta, \) and \( \gamma. \) Reversal of the sign of any one of the direction cosines of the optic axis is equivalent to reversal of the appropriate \( x, y, \) or \( z \) axis. It is clear from Eqs. (5) and (8)--(11) that \( r_\alpha \) and \( r_{pp} \) depend only on the sign of \( \alpha \gamma, \) and the invariance of \( r_\alpha \) and \( r_{pp} \) to the sign of \( \alpha \gamma \) follows from the algebraic identity

\[ \bar{q}^2 = q_o^2 + [\alpha^2]q_o^2 + \beta^2k_o^2 + \gamma^2K^2 + (\alpha\gamma)^2K^2\delta/e_\gamma]\Delta e/e_\gamma. \]  

The numerators and the denominators of \( r_{pp} \) and \( r_\alpha, \) as given by Eqs. (8) and (9) can be split up into invariant (I) and variant (V) parts, namely those that are even and odd in \( \alpha \gamma. \) Both \( r_{pp} \) and \( r_\alpha \) are thus of the form

\[ \frac{P + V'}{I + V} = \left( 1 + V'/V \right) \frac{P}{I} = \left( \frac{I}{1 + V} + \frac{V}{1 + V} \right) V'. \]

For an expression of this type to be invariant we must have \( V'/V = V/I, \) and the value of Eq. (17) is thus \( I/I = V/V. \) When the latter form is used, the \( r_\alpha \) and \( r_{pp} \) reflection amplitudes reduce to

\[ r_\alpha = \frac{(q_1 - q_o)[(q_o + \bar{q})(k_o^2 + q_o\bar{q})] + (\alpha^2k_o^2 + \gamma^2K^2\delta/e_\gamma]\Delta e/e_\gamma)}{(q_1 + q_o)[(q_o + \bar{q})(k_o^2 + q_o\bar{q})] + (\alpha^2k_o^2 + \gamma^2K^2\delta/e_\gamma]\Delta e/e_\gamma} - \frac{\beta^2k_o^2(k_o^2 + q_o\bar{q})\Delta e/e_\gamma}{(q_1 + q_o)[(q_o + \bar{q})(k_o^2 + q_o\bar{q})] + (\alpha^2k_o^2 + \gamma^2K^2\delta/e_\gamma]\Delta e/e_\gamma}, \]

\[ r_{pp} = \frac{(q_1 - q_o)[(q_o + \bar{q})(k_o^2 + q_o\bar{q})] + (\alpha^2k_o^2 + \gamma^2K^2\delta/e_\gamma]\Delta e/e_\gamma)}{(q_1 + q_o)[(q_o + \bar{q})(k_o^2 + q_o\bar{q})] + (\alpha^2k_o^2 + \gamma^2K^2\delta/e_\gamma]\Delta e/e_\gamma} - \frac{\beta^2k_o^2(k_o^2 + q_o\bar{q})\Delta e/e_\gamma}{(q_1 + q_o)[(q_o + \bar{q})(k_o^2 + q_o\bar{q})] + (\alpha^2k_o^2 + \gamma^2K^2\delta/e_\gamma]\Delta e/e_\gamma}. \]

These expressions are manifestly invariant to changes in the sign of the direction cosines. The angle of incidence now appears only through \( K^2 = k_o^2 sin^2 \theta, \) but \( K^2 \) is inside a square root in the variables \( q_1, q_o, \) and \( q_\gamma. \) Consider the effect of changing to the variable \( \epsilon \) defined by

\[ \epsilon_o \omega^2/\epsilon^2 = q_oq_\gamma \quad or \quad \epsilon_1 = q_o/q_1. \]

The variables dependent on the angle of incidence may be written in terms of \( \epsilon: \)

\[ \frac{(cK)}{\omega} = \frac{\epsilon_1(e^2 - e_1e_o)}{\epsilon^2 - e_1^2}, \]

\[ \frac{(c\alpha)}{\omega} = \frac{\epsilon_o(e_\gamma - e_o)}{\epsilon^2 - e_1^2}. \]

Note that \( q_oq_\gamma \) expressed in terms of \( \epsilon \) does not involve a square root. The numerator of \( r_{pp} \) now simplifies to \( (\epsilon + 1)\omega/\epsilon^2 \) times

\[ (e_\gamma - e_o)(\epsilon - e_1)(q_o + q_o)(\bar{q}^2/\omega^2) + \epsilon(e_o - e_1)(\epsilon - e_1) + [\alpha^2\epsilon_\gamma e_\gamma(\epsilon - e_1) + \beta^2e_\gamma(\epsilon - e_1)](\epsilon - e_1) - \gamma^2(\epsilon - e_1)\Delta e/e_\gamma. \]

The remaining square roots are all in the first term. To find the equation determining \( \theta_{pp}, \) we set expression (21) equal to zero, isolate the first term, square, and use Eqs. (6), (20), and

\[ (q_1 + q_o)^2 = \frac{(\epsilon + 1)(e_\gamma - e_o)}{\epsilon - e_1} \frac{\omega^2}{\epsilon - e_1}. \]

The resulting equation is a quartic in \( \epsilon, \) namely,

\[ a_0 + a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + a_4\epsilon^4 = 0, \]

\[ a_0 = -\epsilon_1\epsilon_\gamma(\epsilon(e_\gamma - e_o) - e_1)(\beta^2 + \gamma^2) \]

\[ + [(\beta^2 - \gamma^2)\epsilon_o + \epsilon_\gamma(\epsilon - e_1)]\Delta e/e_\gamma, \]

\[ a_1 = 2\beta^2\epsilon_\gamma(\epsilon(e_\gamma - e_o) - e_1) \]

\[ + \epsilon_\gamma[2\epsilon_\gamma(\epsilon^2 - 1) - 2\gamma^2\epsilon_o]\Delta e/e_\gamma, \]

\[ a_2 = (\epsilon_\gamma - e_o)\epsilon_\gamma(\epsilon(e_\gamma - e_o) - e_1)(\beta^2 + \gamma^2) \]

\[ - \epsilon^2(1 - \gamma^2) + (\epsilon^2 - 1\epsilon^2)\epsilon_\gamma(\beta^2 + \gamma^2) \]

\[ - \epsilon_\gamma(2\epsilon_\gamma(\epsilon(e_\gamma - e_o))\beta^2 + \gamma^2) \]

\[ + \epsilon_\gamma(\epsilon^2 - 1\epsilon^2)\Delta e/e_\gamma, \]

\[ a_3 = 2\beta^2\epsilon_\gamma(\epsilon(e_\gamma - e_o) - e_1) \]

\[ + (\epsilon(e_\gamma - e_o)(\beta^2 + \gamma^2) - e_1)\Delta e/e_\gamma, \]

\[ a_4 = (\epsilon_\gamma - e_o)\epsilon_\gamma(\epsilon^2 - 1\epsilon^2) - \epsilon(e_\gamma - e_o)(\beta^2 + \gamma^2) \]

\[ + (\epsilon(e_\gamma - e_o)(\beta^2 + \gamma^2)\Delta e/e_\gamma. \]

The coefficients \( a_1 \) and \( a_3 \) are zero when the optic axis lies in the plane of incidence (\( \beta = 0 \)). The quartic then reduces to a quadratic in \( \epsilon^2, \) with roots

\[ \frac{\gamma^2e_\gamma e_o}{\epsilon - (1 - \gamma^2)e_\gamma} + \frac{\epsilon_\gamma(\epsilon - e_o)}{\epsilon - e_1}, \]  

\[ \frac{\epsilon_\gamma(\epsilon - e_o)}{\epsilon - e_1}. \]
Only the latter value of $e^2$ is a physical root; the other value does not make expression (21) zero. Since, from Eqs. (20),

$$\tan^2 \theta = \frac{e^2 - e_1 e_0}{e_1(e_0 - e_1)}, \quad (25)$$

the second value of $e^2$ in expression (24) reproduces Eq. (14) and thus also the $\alpha^2 = 1$ and $\gamma^2 = 1$ $\theta_{pp}$ expressions given in Eqs. (15) and (16). When $\beta^2 = 1$, the roots of Eq. (23) are $e_1$, $e_1$, $e_0$, and $e_0$. Again $e = e_1$ does not make expression (21) zero. The (double) physical root is $e_0$, and for this root Eq. (25) gives

$$\tan^2 \theta_{pp}(\beta^2 = 1) = \frac{e_0}{e_1}. \quad (26)$$

(This value also holds when $\alpha^2 e_0 = \gamma^2 e_1$, for any $\beta$.) Thus when the optic axis is normal to the plane of incidence, the zero of $\theta_{pp}$ occurs at the Brewster value for an isotropic medium of refractive index $n_o$, $\theta_B = \arctan(n_o/n_1)$, as is known.

An algorithm for the algebraic solution of a general quartic equation is known, and so it is possible in principle to give an analytic general solution of expression (21). I have been unable to simplify this algebraic solution to a form short enough to be useful or to find factors of ex-

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An algorithm for the algebraic solution of a general quartic equation is known,\textsuperscript{7} and so it is possible in principle to give an analytic general solution of expression (21). I have been unable to simplify this algebraic solution to a form short enough to be useful or to find factors of expression (21) (which would allow the degree of the equation to be reduced). It is possible, however, to give a simple form for the solution to first order in the anisotropy $\Delta e = e_0 - e_1$ by expanding about $e = e_0$, which is the physical root of expression (21) in the isotropic case. The result is

$$e = e_0 + (\alpha^2 e_0 - \gamma^2 e_1) \Delta e/2 e_0. \quad (27)$$

The resulting value of $\tan^2 \theta_{pp}$ as given in Eq. (25), to first order in $\Delta e$, is

$$\tan^2 \theta_{pp} \approx \frac{e_0}{e_1} - \frac{(\alpha^2 e_0 - \gamma^2 e_1) \Delta e}{e_1(e_0 - e_1)}. \quad (28)$$

This approximation gives the exact Brewster angle when $\alpha^2 = 1$ (optic axis coincident with the intersection of the plane of incidence and the reflecting surface), when $\beta^2 = 1$ (optic axis perpendicular to the plane of incidence) and also when $\alpha^2 e_0 = \gamma^2 e_1$, for any $\beta$. Expression (28) is least accurate when $\gamma^2 = 1$ (optic axis normal to the reflecting surface); for calcite in air expression (28) then gives $\theta_{pp} \approx 60.19^\circ$, whereas the exact value from Eq. (16) is $60.72^\circ$, an error of 0.9%. Figure 1 shows the exact $\theta_{pp}$ [Eq. (14)] and the approximate Brewster angles of expression (28) when the optic axis lies in the plane of incidence, for calcite in air.

A better approximation to the Brewster angle is one that contains the $\beta = 0$, $\beta^2 = 1$, and $\alpha^2 e_0 = \gamma^2 e_1$ known solutions (14) and (26) as special cases, namely,

$$\tan^2 \theta_{pp} = \frac{e_0 e_{ay} - e_1 e_y}{e_1(e_0 - e_1)}, \quad (29)$$

where

$$e_{ay} = e_0 + (\alpha^2 + \gamma^2) \Delta e = e_0 - \beta^2 \Delta e. \quad (30)$$

This expression for the Brewster angle would show zero error in Fig. 1, and for calcite in air it has $\theta_{pp}$ errors between $-0.2\%$ and $+0.4\%$.

### 4. EFFECT OF INDEX MATCHING

The reader may have noted that the bounds on the Brewster angle $\theta_{pp}$ given by Eqs. (15) and (16) tend to 0° and 90°, respectively, as $e_1$ increases toward the smaller of $e_0$ and $e_1$. Figure 2 shows the Brewster angle for calcite immersed in a liquid of refractive index 1.48, when the optic axis lies in the plane of incidence. The Brewster angle now ranges from 10.4° to 80.7°, and the error in $\theta_{pp}$ as given by expanding to first order in the anisotropy $\Delta e$, namely expression (28), now ranges to nearly −31%. [The Brewster angle given by expression (29), exact for $\beta = 0$ and $\beta^2 = 1$, now has errors between $-6\%$ and $+5\%$.]

The effect of index matching is thus to enhance the anisotropy, as could already be seen in expression (28), where $\Delta e$ is divided by $e_0 - e_1$. Similar enhancement of anisotropy by index matching has been calculated for optical properties of a uniaxial substrate covered by an isotropic...
layer (for example, the effect of the anisotropy of ice is enhanced a hundredfold by the presence of a water layer).

What happens when the index of the ambient medium lies between the ordinary and the extraordinary indices? Then \( \varepsilon_r - \varepsilon_i \) and \( \varepsilon_r - \varepsilon_s \) have opposite signs, and neither Eq. (15) nor Eq. (16) gives a real Brewster angle. From Eq. (14) we see that, for a zero of \( r_{pp} \) to be possible when the optic axis lies in the plane of incidence, \( \varepsilon_r = \varepsilon_s + \gamma^2 \Delta \varepsilon \) must lie between \( \varepsilon_i \) and \( \varepsilon_r \varepsilon_s / \varepsilon_i \). This range is largest when \( \varepsilon_i \) is close to \( \varepsilon_s \) or \( \varepsilon_r \), and it shrinks to zero when \( \varepsilon_i \) tends to the geometric mean of \( \varepsilon_s \) and \( \varepsilon_r \), i.e., to \( n_r n_s \). When \( \varepsilon_i \) is close to \( n_r n_s \), there is extreme sensitivity to the orientation of the optic axis relative to the normal to the reflecting plane, since for a very small range of \( \gamma^2 \) expression (14) then gives values of \( \theta_{pp} \) ranging from 0° to 90°. Of these, only those smaller than the appropriate critical angle \( \theta_i \) or \( \theta_s \) are possible, since when either \( q_0 \) or \( \overline{q} \) is imaginary the \( r_{pp} \) reflection amplitude is complex, and expressions (14), (15), and (16) no longer apply. The critical angles \( \theta_i \) and \( \theta_s \) are here defined as those angles that make \( q_0 \) and \( \overline{q} \) zero; from Eqs. (4) and (6) we thus have

\[
\sin \theta_i = \frac{n_2}{n_1}, \quad \sin \theta_s = \frac{n_2 \varepsilon_r}{n_1 \varepsilon_s},
\]

(31)

where \( n_{av} \) is the square root of \( \varepsilon_{av} = \varepsilon_r + (\alpha^2 + \beta^2) \Delta \varepsilon \).

When the optic axis lies in the plane of incidence, \( \beta = 0 \) and \( \alpha^2 + \beta^2 = 1 \), so that

\[
\sin \theta_i(\beta = 0) = \frac{n_2}{n_1}.
\]

(32)

From Eq. (14) we have that

\[
\sin^2 \theta_{pp}(\beta = 0) = \frac{\varepsilon_r \varepsilon_i - \varepsilon_r \varepsilon_s}{\varepsilon_r \varepsilon_i - \varepsilon_i^2}.
\]

(33)

Since, as noted above, \( \varepsilon_r \) must lie between \( \varepsilon_i \) and \( \varepsilon_s \varepsilon_i / \varepsilon_r \) for real \( \theta_{pp} \), the condition \( \theta_{pp}(\beta = 0) \leq \theta_i(\beta = 0) \) holds for all \( \gamma \) for which Eq. (14) gives a real Brewster angle. For positive anisotropy \( \Delta \varepsilon = \varepsilon_r - \varepsilon_i > 0 \), the condition \( \theta_{pp} < \theta_s \) is satisfied for

\[
\gamma^2 = \frac{\varepsilon_i \varepsilon_r - \varepsilon_i \varepsilon_s}{\varepsilon_i^2}.
\]

(34)

when \( \varepsilon_i < n_r n_s \), and for \( \gamma^2 < \gamma_s^2 \) when \( \varepsilon_i > n_r n_s \). Thus when \( \varepsilon_r < \varepsilon_i < \varepsilon_s \), negative uniaxial crystals have \( \theta_{pp} \) ranging from 0 to 90°, while positive uniaxial crystals immersed in a medium such that \( \varepsilon_i < \varepsilon_i < \varepsilon_s \) have \( \theta_{pp} \) ranging from 0 to \( \theta_s \). In either case the \( \beta = 0 \) Brewster angle exists only for a range of inclinations of the optic axis to the normal, since the square of the direction cosine \( \gamma \) must lie between

\[
\left| \frac{\varepsilon_r - \varepsilon_i}{\varepsilon_r - \varepsilon_s} \right| \quad \text{and} \quad \left| \frac{\varepsilon_i - \varepsilon_r}{\varepsilon_i - \varepsilon_s} \right|.
\]

(35)

Figure 3 shows \( \theta_{pp}(\beta = 0) \) for calcite immersed in a liquid of refractive index 1.51.

The fact that index matching can extend the range of the Brewster angle from normal to grazing incidence has interesting implications. At grazing incidence \( \theta_0 \to 0 \), and we see from Eqs. (9) and (11) that \( r_{pp} \to 1 \). Thus if \( \theta_{pp} \) is close to 90° there is a conflict of limits, and we may expect a very rapid variation in the \( p \) to \( p \) reflectivity, from zero to unity in a small range near grazing incidence.

One other unexpected phenomenon associated with index matching is the occurrence of two zeros of \( r_{pp} \), at different angles of incidence, for some crystal orientations. The double Brewster angles are associated with \( \theta_i \), becoming complex for \( \theta > \theta_i \). [At \( \theta_i \), \( \overline{q} \) is zero, and for greater angles of incidence \( \overline{q} \) is imaginary; \( \theta_i \) is given by Eq. (31).] Figure 4 shows \( r_{pp} \) for calcite immersed in a medium of refractive index 1.51, with the squares of the direction cosines of the optic axis \( \alpha^2 = 0.02, \beta^2 = 0.67, \gamma^2 = 0.31 \). Beyond \( \theta_i \), the reflection amplitude \( r_{pp} \) is complex, and both the real and the imaginary parts are shown. Note that the second Brewster angle is just a bit less then \( \theta_i \) and that \( |r_{pp}|^2 \) is not unity for \( \theta > \theta_i \) except at grazing incidence.

5. ANGLES AT WHICH \( r_{pp} = 0 \)

An isotropic medium has zero \( r_s \) only when perfectly index matched, in which case \( r_s \) is zero for all angles of incidence.

![Fig. 3. Brewster angles for calcite immersed in a medium of index \( n_s = 1.51 \), for the optic axis in the plane of incidence. Note that a zero of \( r_{pp} \) exists only for a range of inclinations of the optic axis to the surface normal (here between 22° and 66°) as given by expressions (35). These bounding values of the square of the direction cosine \( \gamma \) are shown by vertical dashed lines.](image)

![Fig. 4. Reflection amplitude \( r_{pp} \) for calcite immersed in a liquid of refractive index 1.51 and direction cosines of the optic axis given by \( \alpha^2 = 0.02, \beta^2 = 0.67, \gamma^2 = 0.31 \). Beyond \( \theta_i \) there is an imaginary part of \( r_{pp} \) (dashed curve). The \( \theta_{pp} \) values are 54.03° and 80.27°; the value of \( \theta_i \) is just 1 mdeg more than the larger \( \theta_{pp} \).](image)
We now examine the possibility of zeros of \( r_{ss} \), starting from formulas (18). The procedure and the variables that we use are as for the zeros of \( r_{pp} \), so we need state only the final result, which is that when \( r_{ss} \) is zero a quartic equation in \( \varepsilon = \varepsilon_1 \alpha_0 / q_1 / q_0 \) is satisfied. This quartic is almost identical to the \( r_{pp} \) quartic [Eq. (23)], the difference being that the signs of the coefficients \( a_1 \) and \( a_3 \) are reversed. Now \( a_1 \) and \( a_3 \) are proportional to \( \beta^2 \), and thus solutions of the \( r_{pp} \) and \( r_{ss} \) quartics are the same when \( \beta \) is zero. It is not true, however, that \( r_{ss} \) is zero at the angle given by Eq. (14) when \( \beta = 0 \). This is not a physical root, and in fact when \( \beta = 0 \), \( r_{ss} = (q_0 - q_1)/(q_0 + q_1) \), which is not zero at any angle unless \( n_s = n_p \), in which case it is zero at all angles of incidence. When \( \beta^2 = 1 \) the \( r_{ss} \) quartic factors to \((\varepsilon + \varepsilon_1)^2(\varepsilon + \varepsilon_2)(\varepsilon - \varepsilon_1)\) times a constant, and thus \( r_{ss} \) can be zero only when \( \varepsilon_1 = \varepsilon_2 \) for \( \beta^2 = 1 \), since \( \varepsilon = \varepsilon_0 / q_1 \) must be positive. The \( \alpha^2 = 1 \) and \( \gamma^2 = 1 \) roots are simple but also nonphysical, being a subset of the \( \beta = 0 \) roots.

When \( \alpha^2 \varepsilon_1 = \gamma^2 \varepsilon_1 \), the quartic factors to \((\varepsilon + \varepsilon_1)^2 \) times the quadratic:

\[
a \varepsilon^2 + b \varepsilon + c = 0:
\]

\[
a = (\varepsilon_1^2 - \varepsilon_0^2)[(\varepsilon_2^2 + \beta^2(\varepsilon_0^2 + \varepsilon_1 \varepsilon_2 - \varepsilon_1^2))
+ \varepsilon_2^2(\varepsilon_2 + 2\varepsilon_1 \varepsilon_0 - \varepsilon_1^2 + \varepsilon_1^2 \beta^2)] \Delta \varepsilon,
\]

\[
b = 2\varepsilon_1 \varepsilon_2[\varepsilon_1(2\beta^2 - 1) + \beta^2 \varepsilon_0^2 - \varepsilon_1^2 + (\varepsilon_0 + \beta^2 \varepsilon_1) \Delta \varepsilon],
\]

\[
c = -\varepsilon_2^2(\varepsilon_0^2 - \varepsilon_1^2)(\varepsilon_0^2 + \varepsilon_2^2 + [\varepsilon_0^2(1 - \beta^2 + \beta^4)
+ 2\varepsilon_1 \beta^2(2\beta^2 - 1) + \beta^2 \varepsilon_1^2(2\beta^2 - 3)] \Delta \varepsilon).
\]

The discriminant \( b^2 - 4ac \) is zero when \( \beta^2 = \beta_0^2 \), where

\[
\beta_0^2 = \left(\frac{\varepsilon_1 \varepsilon_2 - \varepsilon_0^2}{2\varepsilon_1^2 \varepsilon_2 - 3\varepsilon_1 \varepsilon_2 - 4\varepsilon_1^2 + \varepsilon_0(3\varepsilon_1 + \varepsilon_2) \Delta \varepsilon}\right). \tag{37}
\]

I have given the details of the \( \alpha^2 \varepsilon_0 = \gamma^2 \varepsilon_1 \) solution because it is the only simple root of the \( r_{ss} = 0 \) quartic known to me and because this configuration will be seen (in Section 6) to give the possibility of simultaneous zeros \( r_{ss} \) and \( r_{pp} \).

Figure 5 shows \( r_{ss} \) as a function of the angle of incidence for the same system that was used to illustrate the existence of double Brewster angles in Fig. 4.

6. SIMULTANEOUS ZERO OF \( r_{pp} \) AND \( r_{ss} \)

Since \( r_{ss} \) as well as \( r_{pp} \) can be zero when the refractive index of the medium of incidence lies beyond the ordinary and the extraordinary indices of the crystal, it is natural to ask whether \( r_{ss} \) and \( r_{pp} \) can be zero at the same angle of incidence. In this section I show that they can (under certain conditions) and that when this happens, one of \( r_{pp} \) or \( r_{ss} \) will also be zero.

We equate the numerators of \( r_{ss} \) and \( r_{pp} \) in Eqs. (18) to zero. From these two simultaneous equations, linear in \( \varepsilon \), we can eliminate \( \varepsilon \) and solve for \( \varepsilon = \varepsilon_0 q_1 / q_0 \). We find that \( \varepsilon \) is determined by an equation linear in \( \varepsilon^2 \) and independent of \( \Delta \varepsilon \), the solution of which is

\[
\varepsilon^2 = \frac{\varepsilon_1 \varepsilon_2(\beta^2 \varepsilon_2 - \gamma^2 \varepsilon_1)}{\varepsilon_1(2\beta^2 - 1) + \varepsilon_0 \alpha^2}. \tag{38}
\]

Both the \( r_{pp} = 0 \) and the \( r_{ss} = 0 \) quartics must be satisfied by this value. Thus we must have, for \( \varepsilon^2 \) as given in Eq. (38),

\[
(a_0 + a_2 \varepsilon_2 + a_4 \varepsilon_1^2) = (a_1 + a_3 \varepsilon_1) \varepsilon^2. \tag{39}
\]

This condition is satisfied by \( \varepsilon_0 = \varepsilon_1 \) and by

\[
\alpha^2 \varepsilon_0 = \gamma^2 \varepsilon_1. \tag{40}
\]

The latter is the physical root, leading to \( \varepsilon = \varepsilon_0 \) and thus from Eq. (25) to

\[
\tan^2 \theta_j = \frac{\varepsilon_0}{\varepsilon_1}. \tag{41}
\]

Thus the joint zero of \( r_{pp} \) and \( r_{ss} \) can occur at one angle of incidence, which is the same as the Brewster angle for an isotropic medium of refractive index \( n_0 \), namely, \( \arctan(n_0 / n_1) \).

The special crystal orientations for which this joint zero can occur follow on substitution of Eq. (40) and its consequence,

\[
\gamma^2 = (1 - \beta^2) \frac{\varepsilon_0}{\varepsilon_1 + \varepsilon_2}, \tag{42}
\]

into the \( r_{ss} = 0 \) quartic. This leads to a linear equation in \( \beta^2 \) and thus to the values

\[
\alpha_1^2 = (\varepsilon_1 + \varepsilon_2)(\varepsilon_0 - \varepsilon_1) / 4 \varepsilon_1 \Delta \varepsilon,
\]

\[
\beta_0^2 = (\varepsilon_1 - \varepsilon_2)[(\varepsilon_1 + \varepsilon_2)^2 + (3\varepsilon_1 + \varepsilon_2) \Delta \varepsilon] / 4 \varepsilon_1^2 \varepsilon_2 \Delta \varepsilon,
\]

\[
\gamma_0^2 = \varepsilon_0 \varepsilon_1(\varepsilon_1 + \varepsilon_2)(\varepsilon_0 - \varepsilon_1) / 4 \varepsilon_1^2 \Delta \varepsilon. \tag{43}
\]

Thus for crystal orientations \( (\pm \alpha_j, \pm |\beta_j|, \pm |\gamma_j|) \) and at angle of incidence \( \theta_j = \arctan(n_0 / n_1) \), \( r_{nn} \) and \( r_{ss} \) will be zero together. For such orientations to exist, each of the squares of the direction cosines given in Eq. (42) must be positive. This will be so only if \( \varepsilon_j \) lies between \( \varepsilon_1 \) and \( \varepsilon_2 \) unless the crystal has strong negative anisotropy (\( \Delta \varepsilon < 0 \)) with

\[
|\Delta \varepsilon| > (\varepsilon_1 + \varepsilon_2)^2 / (3\varepsilon_1 + \varepsilon_2). \tag{44}
\]

(These conditions would make \( \beta_0^2 \) negative.)

Fig. 5. Reflection amplitude \( r_{ss} \) for calcite immersed in a liquid of refractive index 1.51 and direction cosines of the optic axis given by \( \alpha^2 = 0.02, \beta^2 = 0.67, \gamma^2 = 0.31. \) Beyond \( \theta_s = 80.27^\circ \) there is an imaginary part of \( r_{ss} \) (dashed curve). The reflection amplitude is zero at \( \theta_s = 75.12^\circ \).
Finally, note the surprising result that one of \( r_p \) or \( r_s \) will also be zero at \( \theta_j \) for the given crystal orientations. This is because, from Eqs. (20), we see that when \( \epsilon \rightarrow \epsilon_0 \),

\[
\left( \frac{c\eta}{\omega} \right)^2 \rightarrow \frac{\epsilon_1 \epsilon_0}{\epsilon_1 + \epsilon_0}, \quad \left( \frac{c\eta}{\omega} \right)^2 \rightarrow \frac{\epsilon_s^2}{\epsilon_1 + \epsilon_0},
\]

and so \( q_0^2/K^2 \rightarrow \epsilon_s/\epsilon_1 = \gamma_j^2/\alpha_j^2 \), so that, under the conditions of \( r_p \) and \( r_s \) being simultaneously zero, we also have

\[
\alpha^2 q_s^2 - \gamma^2 K^2 = 0. \tag{46}
\]

From Eq. (10) we see that this implies that \( r_p \) will be zero if \( \alpha \) and \( \gamma \) have the same sign and \( r_s \) will be zero if they have opposite signs. Thus three of the four reflection amplitudes will be zero in this configuration. The square of the remaining \( r_p \) or \( r_s \) reflection amplitude then takes the value

\[
\frac{\epsilon_s - \epsilon_1 \epsilon_1 - \epsilon_s}{\epsilon_1^2 - \epsilon_1 \epsilon_s + 3\epsilon_1 \epsilon_s + \epsilon_s \epsilon_s}. \tag{47}
\]

We see that even this can be made zero when the crystal is immersed in a medium with index \( n_s \) or \( n_s' \). For example, when \( \epsilon_1 = \epsilon_s, \alpha_j \) and \( \gamma_j \) are zero, \( \beta_j^2 = 1 \) (optic axis normal to the plane of incidence), and under these conditions all reflection amplitudes are zero at angle of incidence equal to arctan \((n_s/n_s')\).

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