Universal Properties of Electromagnetic Pulses

John Lekner
MacDiarmid Institute and School of Chemistry and Physics
Victoria University, Wellington, New Zealand

Abstract— We survey the existing known universal properties of electromagnetic pulses, and discuss their consequences. The established universal properties are (i) the time invariance of the total electromagnetic energy U, momentum P_z and angular momentum J_z of the pulse, and (ii) the inequality $cP_z < U$. (Pulse propagation is along the z-direction.) In both (i) and (ii), the theorems follow directly from Maxwell's equations.

The conservation of energy, momentum and angular momentum is no surprise, but the inequality $cP_z < U$ implies that all localized electromagnetic pulses have a zero-momentum frame (not a 'rest' frame, waves are never at rest). The above is of course in contradistinction to Einstein's light quantum, for which the momentum $\mathbf{P}$ is purely in one direction, and $cP = U$. The implication seems to be that we cannot form a model of the photon by any pulse wave-function satisfying Maxwell's equations. If the momentum $\mathbf{P}$ and energy U formed a four-vector $(c\mathbf{P}, U)$, $U^2 - c^2P^2$ would be a Lorentz invariant. This holds for point particles, but not universally for wavepackets. We show however that $u^2 - c^2p^2$ is a Lorentz invariant, non-negative at all space-time points (u and $\mathbf{p}$ are the energy and momentum densities).

We also discuss the helicity of electromagnetic pulses, and the counter-intuitive relation between the helicity and angular momentum of certain exactly calculable examples.

1. INTRODUCTION

Maxwell's equations, with the electric and magnetic fields expressed in terms of the vector potential $\mathbf{A}(\mathbf{r}, t)$ and scalar potential $\Phi(\mathbf{r}, t)$ via

$$\mathbf{E} = -\nabla\Phi - \partial_{ct}\mathbf{A}, \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{1}$$

and with $\mathbf{A}$ and Φ satisfying the Lorenz condition $\nabla \cdot \mathbf{A} + \partial_{ct}\Phi = 0$, lead (in free space) to Φ and all components of $\mathbf{A}$ satisfying the wave equation

$$\nabla^2\psi - \partial_{ct}^2\psi = 0 \tag{2}$$

Electromagnetic pulses can then be constructed from solutions of (2). For example, the choice $\Phi = 0$, $\mathbf{A} = \nabla \times (0, 0, \psi) = (\partial_y, -\partial_x, 0)\psi$ gives us the transverse electric (TE) pulse with

$$\mathbf{E} = -\partial_{ct}\mathbf{A} = (-\partial_y\partial_{ct}, \partial_x\partial_{ct}, 0)\,\psi, \qquad \mathbf{B} = \nabla \times \mathbf{A} = \left(\partial_x\partial_z, \partial_y\partial_z, -\partial_x^2 - \partial_y^2\right)\psi \tag{3}$$

The wave Equation (2) has an infinity of solutions, for example $\psi = f(z - ct)$, with f an arbitrary twice differentiable function. These solutions, and the textbook plane wave $\exp i(\mathbf{K} \cdot \mathbf{r} - \omega t)$ and spherical waves $r^{-1}\exp[\pm iK(r \pm ct)]$ (both with $\omega = cK$) are not localized in space-time. The spherical wave solutions generalize to $r^{-1}f(r \pm ct)$, with f again any twice-differentiable function. These solutions are singular at the origin.

Bateman [1] obtained a general solution of the wave equation in integral form. For solutions with axial symmetry (independent of the azimuthal angle ϕ) the Bateman solution is, with $\rho = [x^2 + y^2]^{\frac{1}{2}}$ being the distance from the z-axis,

$$\psi(\rho, z, t) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, f(z + i\rho\cos\theta, ct + \rho\sin\theta) \tag{4}$$

We outline a proof (different from Bateman's): the wave equation in cylindrical polars, with no azimuthal dependence, reads

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2 - \partial_{ct}^2\right)\psi = 0 \tag{5}$$

Carrying out the partial differentiations in $(\nabla^2 - \partial_{ct}^2)f$, and comparing with $\partial_\theta^2 f$ gives us

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2 - \partial_{ct}^2\right)f = -\rho^{-2}\partial_\theta^2 f \tag{6}$$

Operating on (4) with $\nabla^2 - \partial_{ct}^2$ therefore gives

$$-2\pi\rho^2\left(\nabla^2 - \partial_{ct}^2\right)\psi = \int_0^{2\pi} d\theta\, \partial_\theta^2 f = \partial_\theta f\,\Big|_0^{2\pi} = 0 \tag{7}$$

On the propagation axis ($\rho = 0$) the beam wavefunction becomes

$$\psi(0, z, t) = f(z, ct) \tag{8}$$

For example, if the on-axis wavefunction takes the form

$$f(z,t) = \frac{ab}{[a - i(z+ct)]\,[b + i(z-ct)]}\psi_0 \tag{9}$$

the corresponding full wavefunction obtained by integrating (4) is

$$\psi(\rho, z, t) = \frac{ab}{\rho^2 + [a - i(z+ct)]\,[b + i(z-ct)]}\psi_0 \tag{10}$$

This wavefunction has been obtained by other means (see references in [3–6]).

2. CONSERVATION LAWS, ENERGY-MOMENTUM INEQUALITIES

The energy, momentum and angular momentum densities of an electromagnetic field, in free space and in Gaussian units, are [2]

$$u(\mathbf{r},t) = \frac{1}{8\pi}\left(E^2 + B^2\right), \qquad \mathbf{p}(\mathbf{r},t) = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}, \qquad \mathbf{j}(\mathbf{r},t) = \mathbf{r}\times\mathbf{p}(\mathbf{r},t) \tag{11}$$

$\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are the real electric and magnetic fields at position $\mathbf{r}$ and time t. The total energy, momentum and angular momentum at time t of an electromagnetic pulse are

$$U = \int d^3r\, u(\mathbf{r},t), \qquad \mathbf{P} = \int d^3r\, \mathbf{p}(\mathbf{r},t), \qquad \mathbf{J} = \int d^3r\, \mathbf{j}(\mathbf{r},t) \tag{12}$$

It will come as no surprise that these are all conserved quantities: the integrals in (12) are all independent of time.

The energy and momenta of electromagnetic pulses based on the solution (10) of the wave equation were evaluated in [3]. Proofs of the constancy of U and of $\mathbf{P}$ were sketched in [4]. The conservation of angular momentum was proved in [5]. In all cases, the proofs follow from taking the time derivatives of the quantities U, $\mathbf{P}$ and $\mathbf{J}$ defined in (12), applying Maxwell's free-space equations

$$\begin{array}{cc} \nabla\cdot\mathbf{B} = 0 & \nabla\cdot\mathbf{E} = 0 \\ \nabla\times\mathbf{E} + \partial_{ct}\mathbf{B} = 0 & \nabla\times\mathbf{B} - \partial_{ct}\mathbf{E} = 0 \end{array} \tag{13}$$

and using elementary analytical techniques.

In order for the quantities U, $\mathbf{P}$ and $\mathbf{J}$ to *exist* (let alone be conserved), the corresponding electromagnetic pulse has to be localized. The first evaluation of U for any localized pulse was in [6]; later evaluation of energy, momentum and angular momentum for various electromagnetic pulses found [3] that all had $U > cP_z$, with the transverse momenta P_x and P_y zero. Thus these pulses could be Lorentz-transformed into their zero momentum frames, in which the pulse converges onto its focal region and then diverges from it, maintaining zero net momentum at all times. The proof that $U > cP_z$ for all localized electromagnetic pulses is elementary [4]: let the total momentum vector $\mathbf{P}$ point along the z direction, and consider the energy and momentum densities $u(\mathbf{r},t)$ and $p_z(\mathbf{r},t)$. From (11), we have

$$\begin{aligned} 8\pi(u - cp_z) &= \mathbf{E}^2 + \mathbf{B}^2 - 2(\mathbf{E}\times\mathbf{B})_z \\ &= E_x^2 + E_y^2 + E_z^2 + B_x^2 + B_y^2 + B_z^2 - 2(E_xB_y - E_yB_x) \\ &= (E_x - B_y)^2 + (E_y + B_x)^2 + E_z^2 + B_z^2 \geq 0 \end{aligned} \tag{14}$$

Equality of U and cP_z would require $u - cp_z$ to be zero everywhere and at all times, which from (14) requires $E_z = 0 = B_z$ (purely transverse fields) and also $E_x = B_y$ and $E_y = -B_x$. The divergence equations in (13) then give

$$-\partial_x E_y + \partial_y E_x = 0 \qquad \text{and} \qquad \partial_x E_x + \partial_y E_y = 0 \tag{15}$$

Thus E_x and $-E_y$ would be a Cauchy-Riemann pair in the variables x and y, and satisfy

$$\left(\partial_x^2 + \partial_y^2\right) E_x = 0, \qquad \left(\partial_x^2 + \partial_y^2\right) E_y = 0 \tag{16}$$

Such harmonic functions cannot have a maximum except at the boundary of their domain, and thus cannot be localized in x and y (for any z and t). For localized electromagnetic pulses we therefore always have the total energy greater than c times the net total momentum

$$U > cP_z \tag{17}$$

U and $\mathbf{P}$ are defined by (12) as spatial integrals, independent of time in any given inertial frame. If together they formed the four-vector $(c\mathbf{P}, U)$, $U^2 - c^2P^2$ would be a Lorentz invariant, the same in all inertial frames.

Such four-vectors exist for point particles, but cannot be associated (in general) with extended wavepackets. Consider however the squares of the volume densities, $u^2(\mathbf{r}, t)$ and $\mathbf{p}^2(\mathbf{r}, t)$. From (11), we have

$$\begin{aligned} (8\pi)^2(u^2 - c^2\mathbf{p}^2) &= \left(E^2 + B^2\right)^2 - 4\left(\mathbf{E} \times \mathbf{B}\right)^2 \\ &= \left(E^2 + B^2\right)^2 - 4E^2B^2 + 4(\mathbf{E} \cdot \mathbf{B})^2 \\ &= \left(E^2 - B^2\right)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2 \end{aligned} \tag{18}$$

Hence $u^2 - c^2\mathbf{p}^2$ is everywhere non-negative, and further it is a Lorentz invariant, since $E^2 - B^2$ and $\mathbf{E} \cdot \mathbf{B}$ are Lorentz invariants. The Appendix has further discussion of the Lorentz transformation of wavepackets.

3. ANGULAR MOMENTUM, HELICITY

We have seen that the energy U, momentum $\mathbf{P}$ and angular momentum $\mathbf{J}$ are all conserved (do not change with time) for any electromagnetic pulse. The energy and momentum are also independent of the choice of origin of the spatial coordinates (which are integrated over, see (12)). However, the angular momentum does depend on the choice of origin: in the translation $\mathbf{r} \to \mathbf{r} - \mathbf{a}$, $\mathbf{J} \to \mathbf{J} - \mathbf{a} \times \mathbf{P}$. Textbooks make statements such as ([7], p569) 'the photon has vanishing mass and cannot be brought to rest in any Lorentz frame of reference'. As we have seen, any localized electromagnetic pulse satisfying Maxwell's equations does have a zero momentum frame (not a 'rest' frame). In the frame where $\mathbf{P}$ is zero the angular momentum is independent of the choice of origin, and thus we can associate an *intrinsic angular momentum* with a localized electromagnetic pulse.

Suppose (as we have in this paper) that the net momentum of a pulse is along the z-direction, $\mathbf{P} = (0, 0, P_z)$. A Lorentz boost at speed c^2P_z/U, along the z-axis, will bring the pulse to its zero momentum frame. The component J_z of the angular momentum is unchanged in this Lorentz transformation. This is because the four-tensor of angular momentum $J_{ij} = X_iP_j - X_jP_i$ (X_i and P_i represent components of the space-time and momentum-energy four-vectors) has the same structure as the electromagnetic field four-tensor composed of $\mathbf{E}$ and $\mathbf{B}$ ([8], Section 2–6)

$$[J_{ij}] = \begin{pmatrix} 0 & J_z & -J_y & J_{14} \\ -J_z & 0 & J_x & J_{24} \\ J_y & -J_x & 0 & J_{34} \\ J_{41} & J_{42} & J_{43} & 0 \end{pmatrix} \tag{19}$$

where

$$\begin{aligned} J_{41} = -J_{14} = i(ctP_x - xU/c) \\ J_{42} = -J_{24} = i(ctP_y - yU/c) \\ J_{43} = -J_{34} = i(ctP_y - zU/c) \end{aligned} \tag{20}$$

For comparison, the field four-tensor, also in the Minkowski notation, is

$$[F_{ij}] = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix} \tag{21}$$

Since B_z is unchanged by a Lorentz boost along the z-axis, J_z will also be unchanged by such a transformation. Thus we can regard the component of the angular momentum along the momentum (J_z, in this paper) as the intrinsic angular momentum of the pulse.

The helicity of the pulse is $+1$ if the sign of J_z is the same as that of P_z (in a frame with $P_z \neq 0$), -1 if the signs are opposite. There is no helicity (or the helicity is zero) if J_z is zero.

We shall give some examples of results for electromagnetic pulses based on the wavefunction (10). The first is for the TE+iTM pulse for which

$$\mathbf{A} = \nabla \times (0, 0, \psi) = (\partial_y, -\partial_x, 0)\psi \tag{22}$$

$$\mathbf{B} = \nabla \times \mathbf{A} + i\partial_{ct}\mathbf{A}, \qquad \mathbf{E} = i\mathbf{B} \tag{23}$$

(Here $\mathbf{B}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ are complex; their real and imaginary parts are separately solutions of Maxwell's equations.) The energy, momentum and angular momentum found in [3] are

$$U = \frac{\pi}{8}\frac{a+b}{ab}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{a-b}{ab}\psi_0^2, \quad J_z = 0 \tag{24}$$

For this pulse, a Lorentz boost at speed βc, $\beta = cP_z/U = (a-b)/(a+b)$, will bring the pulse to its zero-momentum frame [3].

If instead we take the vector potential to be

$$\mathbf{A} = \nabla \times [i\psi, \psi, 0] \tag{25}$$

with $\mathbf{B}$ and $\mathbf{E}$ defined by (23) as before, we obtain [3]

$$U = \frac{\pi}{8}\frac{a+3b}{a^2}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{a-3b}{a^2}\psi_0^2, \qquad cJ_z = \frac{\pi}{4}\frac{b}{a}\psi_0^2 \tag{26}$$

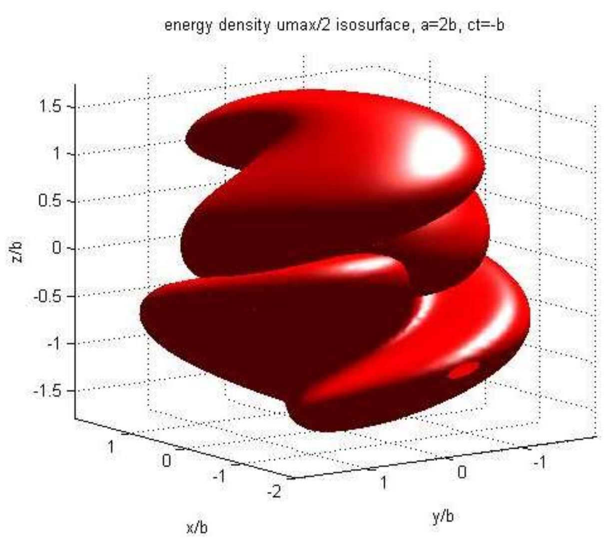

Figure 1: The energy density isosurface at $u = \frac{1}{2}u_{\max}$, $ct = -b$ for the wavefunction given in the text. The pulse is travelling upward, and has negative angular momentum about the propagation direction. In contrast, the energy isosurface consists of two short right-handed screw threads.

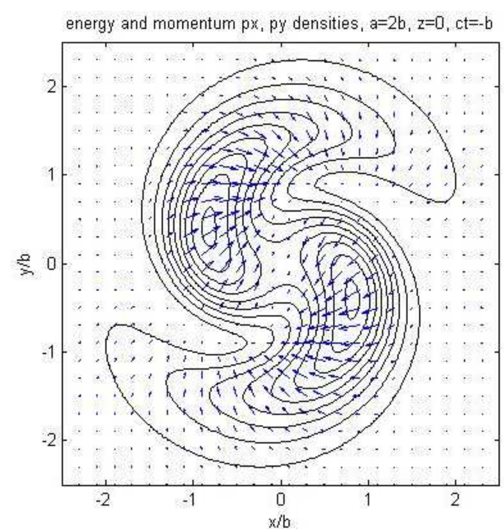

Figure 2: The energy density (contours) and the transverse momentum densities p_x, p_y (arrows) in the $z = 0$ plane, for the same pulse and at the same time as in Figure 1. The pulse is travelling up out of the page.

This example shows that non-zero angular momentum can result from a wavefunction without azimuthal dependence: the curl operator supplies the twist.

More complex exact solutions of the wave equation have been tried, and the energy, momentum and angular momentum evaluated [9, 10]. There we find the surprising result that when the wavefunction ψ has an $e^{im\phi}$ azimuthal dependence, the helicity is opposite to the sign of m. Since J_z is represented by the operator $-i\hbar\partial_\phi$ in quantum mechanics, $J_z e^{im\phi} = \hbar m e^{im\phi}$, so there the $e^{im\phi}$ dependence produces $J_z = \hbar m$, the same sign as m. It is not understood physically why electromagnetic pulses do the opposite.

Figure 1 illustrates a pulse based on ψ equal to $\rho\, e^{i\phi}/[b + i(z - ct)]$ times the wavefunction in (10), with $\mathbf{A}$ given by (22) and $\mathbf{E}$ and $\mathbf{B}$ by (1) (with Φ zero). The resulting energy, momentum and angular momentum are [10]

$$U = \frac{\pi}{16}\frac{3a+b}{b^2}\psi_0^2, \qquad cP_z = \frac{\pi}{16}\frac{3a-b}{b^2}\psi_0^2, \qquad cJ_z = -\frac{\pi}{8}\frac{a}{b}\psi_0^2 \tag{27}$$

Note that energy isosurface has positive helicity (right-handed), opposite to that of the angular momentum.

4. DISCUSSION

The established universal properties of localized electromagnetic pulses are the constancy of their energy, momentum and angular momentum in time, and the fact that their energy is always greater then c times their momentum. As a consequence, localized electromagnetic pulses have a zero-momentum frame. A further consequence is that we can define an intrinsic angular momentum for such pulses.

Localized solutions of the classical Maxwell equations thus stand in contradistinction to Einstein's light quantum [11], for which $U = cP$, and which cannot be transformed to a zero momentum frame.

APPENDIX: LORENTZ TRANSFORMATION OF WAVEPACKETS

For point particles of mass M, the energy and momentum are related by $U^2 = M^2c^4 + P^2c^2$, and the combination $(c\mathbf{P}, U)$ is a four-vector, meaning that it transforms in the same way as $(\mathbf{r}, ct)$. It follows that $U^2 - c^2P^2$ is a Lorentz invariant, in this case M^2c^4.

Electromagnetic wavepackets are extended objects, evolving in space-time, and the transformation between inertial frames is more complicated. However, as we have seen in Equation (18), $u^2 - c^2p^2$ is a non-negative Lorentz invariant, for any electromagnetic pulse.

Consider the transformation of a scalar wavefunction such as (10). A Lorentz boost along the direction of motion (i.e. along the z-axis) at speed βc leaves the transverse coordinate ρ unchanged, and changes z and t to z' and t':

$$z = \gamma\left(z' + \beta ct'\right), \qquad ct = \gamma\left(ct' + \beta z'\right), \qquad \gamma = (1 - \beta^2)^{-\frac{1}{2}} \tag{A1}$$

The effect is to change the weight of the $z \pm ct$ components of ψ:

$$z + ct = \sqrt{\frac{1+\beta}{1-\beta}}\left(z' + ct'\right), \qquad z - ct = \sqrt{\frac{1-\beta}{1+\beta}}\left(z' - ct'\right) \tag{A2}$$

For the wavefunction in (10), a Lorentz boost with $\beta = (a - b)/(a + b)$ or $(1 + \beta)/(1 - \beta) = a/b$ transforms ψ to [3]

$$\psi\left(\mathbf{r}', t'\right) = \frac{ab\psi_0}{\rho^2 + \left[\sqrt{ab} - i\left(z' + ct'\right)\right]\left[\sqrt{ab} + i\left(z' - ct'\right)\right]} \tag{A3}$$

in which the forward and backward propagations are balanced. Such a choice of β brings the TE + iTM pulse to its zero momentum frame, as we have seen in Equations (22) to (24). Moreover, the energy in the zero momentum frame, $U_0 = \frac{\pi}{4}\psi_0^2/\sqrt{ab}$, is equal to the square root of $U^2 - c^2P_z^2$, so in this respect the pulse momentum and energy behave as four-vector components.

However, other pulses constructed from the same wavefunction require a different β to bring them to their zero momentum frame, as in the example specified by (25) and (26) for which $\beta = (a - 3b)/(a + 3b)$. For this β the wavefunction (10) is transformed to

$$\psi\left(\mathbf{r}', t'\right) = \frac{ab\psi_0}{\rho^2 + \left[\sqrt{ab} - i/\sqrt{3}\left(z' + ct'\right)\right]\left[\sqrt{ab} + i\sqrt{3}\left(z' - ct'\right)\right]} \tag{A4}$$

The transformed momentum is zero, and the transformed energy is

$$U_0 = \frac{\pi}{4}\psi_0^2/\sqrt{3ab} \tag{A5}$$

This is not (unless $a = 3b$) equal to the square root of $U^2 - c^2 P_z^2$, for which the values in (26) give

$$\sqrt{U^2 - c^2 P_z^2} = \frac{\pi}{4}\psi_0^2 \sqrt{\frac{3b}{a^3}} \tag{A6}$$

Thus the same solution of the wave equation can lead to pulses for which the energy and momenta may or may not behave like four-vectors. In general, the Lorentz transformation of electromagnetic wavepackets is more complicated than that of point particles, as may be expected.

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