Universal Properties of Electromagnetic Beams

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Abstract— We survey the existing known universal properties of electromagnetic beams, and discuss the possibility of a new one. The established universal properties are the existence of beam invariants, which follow from conservation laws, and various non-existence theorems arising out of beam localization transversely to the direction of propagation. In both cases the theorems follow directly from Maxwell's equations. The conservation of energy, momentum and angular momentum lead to seven beam invariants, the simplest of which is the integral across a transverse section of an electromagnetic beam of the time-averaged longitudinal component of the momentum density. This integral is the same for any transverse section: it is an *invariant*. This particular invariant comes from the conservation of energy; the conservation of momentum leads to three invariant integrals over components of the electromagnetic stress tensor, and likewise three more invariants come from the conservation of angular momentum. The non-existence theorems show that the properties of the textbook electromagnetic plane wave cannot be realized for real (transversely finite) beams. They assert the non-existence of: (i) Pure TEM beam modes, (ii) Beams of fixed linear polarization, (iii) Beams which are everywhere circularly polarized in a fixed plane, and (iv) Beams within which the energy velocity is everywhere in the same direction and magnitude c. In addition to these established facts, we give an elementary topological argument for the universality of rings of zeros of the beam wave function in the focal plane, leading to wave vortices.

1. INTRODUCTION

This paper is concerned with *universal* properties of electromagnetic beams, by which we mean properties that all physical beams must have (or cannot have). It is useful however to have a summary of the existing exact solutions of Maxwell's equations representing electromagnetic beams, which we give in this introduction.

For monochromatic beams, in which the time dependence of the complex fields is contained in the factor $e^{-i\omega t}$, all the components of $\mathbf{E}$ and $\mathbf{B}$ satisfy the Helmholtz equation,

$$(\nabla^2 + K^2)\psi = 0, \qquad K = \omega/c \tag{1}$$

This follows from Maxwell's equations by expressing the magnetic and electric fields in terms of the vector and scalar potentials $\mathbf{A}$ and Φ,

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\Phi - \partial_{ct}\mathbf{A} \tag{2}$$

and choosing the Lorenz gauge

$$\nabla \cdot \mathbf{A} + \partial_{ct}\Phi = 0 \tag{3}$$

In free space, Φ and all components of $\mathbf{A}$ satisfy (1) (see for example [1, pp.218ff]), and so do their derivatives such as $\mathbf{E}$ and $\mathbf{B}$.

The textbook solutions of (1) are the plane wave $\exp(i\mathbf{K}\cdot\mathbf{r})$ and the spherical waves $\exp(\pm iKr)/r$. Physical beams are localized transversely to the direction of propagation, in contradistinction to the textbook solutions. Deschamps [2] noted that a complex shift along the propagating direction (the z-axis, in this paper) gives an exact solution of (1) localized transversely:

$$\psi = \frac{e^{ikR}}{R}, \qquad R^2 = x^2 + y^2 + (z - ib)^2 = \rho^2 + (z - ib)^2 \tag{4}$$

This solution is singular on the circle $\{\rho = b,\, z = 0\}$, and so cannot represent a physical beam. One can regularize by subtracting the complex-shifted spherically converging wave $\exp(-iKR)/R$ [3]

$$\psi = \frac{\sin KR}{KR} = j_0(KR) \tag{5}$$

and generalize to [4]

$$\psi_{\ell m} = j_\ell(KR)\, P_\ell^m\left(\frac{z-ib}{R}\right) e^{im\phi} \tag{6}$$

but problems remain in the divergence of some invariants (see [5] and Section 3 below), and in the backward-propagating components associated with the terms proportional to $\exp(-iKR)/R$.

The Helmholtz Equation (1) is separable in cylindrical coordinates (ρ, ϕ, z): it reads

$$\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\phi^2 + \partial_z^2 + K^2\right] \psi = 0 \tag{7}$$

This is solved by $J_m(k\rho)e^{im\phi}e^{iqz}$ provided $k^2 + q^2 = K^2$, and thus also by the *generalized Bessel beams* [6]

$$\psi_m(\mathbf{r}) = e^{im\phi}\int_0^K dk\, f(k)\, J_m(k\rho)e^{iqz}, \qquad k^2+q^2 = K^2 \tag{8}$$

Note that k is restricted to the interval $[0, K]$, so $q = \sqrt{K^2 - k^2}$ is a real mapping onto $[0, K]$. These beams are purely forward propagating, by construction. The weight function $f(k)$ can be complex; it is constrained by the necessary finiteness of integral invariants (Section 3). In the next section we shall see how $f(k)$ is related to Bateman's weight function.

2. BATEMAN INTEGRAL SOLUTION OF THE WAVE EQUATION

Bateman [7] considered integral representations of solutions to the wave equation $(\nabla^2 - \partial_{ct}^2)\psi = 0$. The simplest case is that where the solution is independent of the azimuthal angle ϕ, in which case it takes the form

$$\Psi(\rho, z, t) = \frac{1}{2\pi}\int_0^{2\pi} d\theta F(z + i\rho\cos\theta,\, ct + \rho\sin\theta) \tag{9}$$

We can adapt this to find the general solution of the Helmholtz Equation (1) which is independent of the azimuthal angle. For time-dependence $e^{-i\omega t} = e^{-iKct}$, the function F must take the form

$$F(z + i\rho\cos\theta, ct + \rho\sin\theta) = g(z + i\rho\cos\theta)\, e^{-iK(ct+\rho\sin\theta)} \tag{10}$$

and then the spatial part of Ψ in (9) becomes

$$\psi(\rho, z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, g(z + i\rho\cos\theta)\, e^{-iK\rho\sin\theta} \tag{11}$$

We can verify that this is a solution of the Helmholtz equation as follows. Let $G(\rho, z, \theta) = g(z + i\rho\cos\theta)\, e^{-iK\rho\sin\theta}$. A short calculation shows that $(\nabla^2+K^2)G = -\rho^{-2}\partial_\theta^2 G$ and so $2\pi(\nabla^2+K^2)\psi = -\rho^{-2}\partial_\theta G|_0^{2\pi} = 0$. Thus the $\psi(\rho, z)$ of (11) is the most general form of the scalar wavefunction corresponding to axially symmetric monochromatic beams.

Note that on the beam axis $(\rho = 0)$ we get

$$\psi(0, z) = g(z) \tag{12}$$

Thus the amplitude function g in (11) given by the axial value of the beam wavefunction.

There is a one-to-one correspondence between (11) and the $m = 0$ generalized Bessel beam solution (8). Since $k^2 + q^2 = K^2$ we can write $\psi_0(\mathbf{r})$ as an integral over q instead of over k:

$$\psi_0(\mathbf{r}) = \int_0^K dq\, h(q) J_0\left(\rho\sqrt{K^2-q^2}\right) e^{iqz} \qquad h(q) = \frac{q}{\sqrt{K^2-q^2}} f\left(\sqrt{K^2-q^2}\right) \tag{13}$$

The zero-order Bessel function containing the square root can be rewritten by using Bessel's integral ([8, Section 2.21]), which transforms (13) into

$$\psi_0(\rho, z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-iK\rho\sin\theta}\int_0^K dq\, h(q) e^{iq(z+i\rho\cos\theta)} \tag{14}$$

Comparison of (11) and (14) shows that, for the $m = 0$ generalized Bessel beams, the amplitude function g is given by

$$g(z + i\rho\cos\theta) = \int_0^K dq\, h(q) e^{iq(z+i\rho\cos\theta)} \tag{15}$$

The axial form of ψ is thus equal to the finite Fourier transform of $h(q)$:

$$\psi(0, z) = g(z) = \int_0^K dq\, h(q) e^{iqz} \tag{16}$$

3. CONSERVATION LAWS AND BEAM INVARIANTS

The energy, momentum and angular momentum densities of an electromagnetic field in free space are, in Gaussian units, given by [1]

$$u(\mathbf{r}, t) = \frac{1}{8\pi}(E^2 + B^2), \qquad \mathbf{p}(\mathbf{r}, t) = \frac{1}{4\pi c}\mathbf{E} \times \mathbf{B}, \qquad \mathbf{j}(\mathbf{r}, t) = \mathbf{r} \times \mathbf{p}(\mathbf{r}, t) \tag{17}$$

Here $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ are the real fields. For monochromatic fields it is convenient to work in terms of complex fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ with the real electric field being given by

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r})e^{-i\omega t}\} = \mathrm{Re}\{[\mathbf{E}_r(r) + i\mathbf{E}_i(\mathbf{r})][\cos\omega t - i\sin\omega t]\} = \mathbf{E}_r(\mathbf{r})\cos\omega t + \mathbf{E}_i(\mathbf{r})\sin\omega t \tag{18}$$

The average of $u(\mathbf{r}, t)$ over one period $2\pi/\omega$ is

$$\bar{u}(\mathbf{r}) = \frac{1}{8\pi}\{\mathbf{E}(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r}) + \mathbf{B}(\mathbf{r}) \cdot \mathbf{B}^*(\mathbf{r})\} \tag{19}$$

Likewise the cycle-averaged momentum density is

$$\bar{\mathbf{p}}(\mathbf{r}) = \frac{1}{16\pi c}[\mathbf{E}(\mathbf{r}) \times \mathbf{B}^*(\mathbf{r}) + \mathbf{E}^*(\mathbf{r}) \times \mathbf{B}(\mathbf{r})] \tag{20}$$

The conservation of energy equation, $\nabla \cdot \mathbf{S} + \partial_t u = 0$, where $\mathbf{S} = c^2\mathbf{p}$ is the energy flux density, has the cycle-average

$$\nabla \cdot \bar{\mathbf{p}} = \partial_x \bar{p}_x + \partial_y \bar{p}_y + \partial_z \bar{p}_z = 0 \tag{21}$$

Applying $\int d^2r = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy = \int_0^{\infty} d\rho\rho \int_0^{2\pi} d\phi$ to (21) gives, for transversely finite beams propagating in the z direction [9]

$$\partial_z \int d^2r\, \bar{p}_z = 0, \qquad \text{or} \qquad P'_z = \int d^2r\, \bar{p}_z = \text{constant} \tag{22}$$

We use the notation P'_z, since $dP'_z = P'_z dz$ is total z-component momentum contained in a transverse slice of the beam, of thickness dz. Equation (22) states that the momentum content per unit length, along the direction of net propagation of the beam, is an *invariant*. Note that the invariance of the *momentum* content per unit length is derived from the conservation of *energy*.

The conservation of momentum equation is expressed in terms of the stress (or momentum flux density) tensor

$$\partial_t p_i + \sum_j \partial_j \tau_{ij} = 0, \qquad \tau_{ij} = \frac{1}{4\pi}\left[\frac{1}{2}(E^2 + B^2)\delta_{ij} - E_iE_j - B_iB_j\right] \tag{23}$$

Taking the cycle average gives $\sum_j \partial_j \bar{\tau}_{ij} = 0$, and operating with $\int d^2r$ gives $\partial_z \int d^2r \bar{\tau}_{iz} = 0$ ($i = x, y, z$). Thus momentum conservation leads to three invariants [9]

$$\begin{aligned}
T'_{xz} &= \int d^2r\, \bar{\tau}_{xz} = -\frac{1}{4\pi}\int d^2r\, \left[\overline{E_xE_z} + \overline{B_xB_z}\right] \\
T'_{yz} &= \int d^2r\, \bar{\tau}_{yz} = -\frac{1}{4\pi}\int d^2r\, \left[\overline{E_yE_z} + \overline{B_yB_z}\right] \\
T'_{zz} &= \int d^2r\, \bar{\tau}_{zz} = \frac{1}{8\pi}\int d^2r\, \left[\overline{E_x^2 + E_y^2 - E_z^2} + \overline{B_x^2 + B_y^2 - B_z^2}\right]
\end{aligned} \tag{24}$$

Three more invariants follow from the conservation of angular momentum, $\partial_t j_i + \sum_\ell \partial_\ell \mu_{\ell i} = 0$, where the angular momentum flux density tensor $\mu_{\ell i} = \sum_j \sum_k \varepsilon_{ijk} x_j \tau_{k\ell}$ is defined in terms of the momentum flux density tensor τ_{ij} [10]. These invariants are [9]

$$\begin{aligned} M'_{zx} &= \int d^2r\, \bar{\mu}_{zx} = \int d^2r [y\bar{\tau}_{zz} - z\bar{\tau}_{yz}] \\ M'_{zy} &= \int d^2r\, \bar{\mu}_{zy} = \int d^2r [z\bar{\tau}_{xz} - x\bar{\tau}_{zz}] \\ M'_{zz} &= \int d^2r\, \bar{\mu}_{zz} = \int d^2r [x\bar{\tau}_{yz} - y\bar{\tau}_{xz}] \end{aligned} \tag{25}$$

Thus there are seven universal invariants of electromagnetic beams, arising from the conservation of energy, momentum and angular momentum. Perhaps surprisingly, the energy per unit length of the beam, $U' = \int d^2r\, \bar{u}$, is not always an invariant, although it is constant for the types of generalized Bessel beams discussed in [6], as is $J'_z = \int d^2r \bar{j}_z$.

4. NON-EXISTENCE THEOREMS

In textbooks a light beam is usually represented by a plane wave, with **E**, **B** and the propagation vector **k** everywhere mutually perpendicular. This 'beam' can be everywhere linearly polarized in the same direction, or everywhere circularly polarized in the same plane, and its energy is everywhere transported in a fixed direction at the speed of light. It has been shown in [11] that *none* of these properties can hold for a transversely finite beam. We shall just state the theorems, except for the one relating to linear polarization, for which the proof in [11] is incomplete.

(i) Pure TEM beams do not exist.
(ii) Beams of fixed linear polarization do not exist.
(iii) Beams which are everywhere circularly polarized in the same direction do not exist.
(iv) Beams or pulses within which the energy velocity [12] is everywhere in the same direction and of magnitude c do not exist.

Proof of (ii): Suppose $\mathbf{E} = (F(x, y, z), 0, 0)$, so the beam is linearly polarized along $\hat{\mathbf{x}}$, everywhere. Then from the Maxwell curl equations, with e^{-iKct} time dependence, we have $iK\mathbf{B} = \nabla \times \mathbf{E} = (0, \partial_z, -\partial_y)F$, and $iK\nabla \times \mathbf{B} = \left(-[\partial_y^2 + \partial_z^2], \partial_x\partial_y, \partial_x\partial_z\right) F = K^2\mathbf{E}$. Hence $(\partial_y^2 + \partial_z^2 + K^2)F = 0$ and $\partial_x\partial_y F = 0 = \partial_x\partial_z F$. The last two equations imply $F(x, y, z) = f(x) + g(z) + h(y, z)$, which cannot represent a beam localized transversely in the x direction.

5. FOCAL PLANE ZEROS, AND DISCUSSION

We have seen that electromagnetic beams can be constructed from solutions of the scalar Helmholtz Equation (1). In particular the TM, TE, 'LP' and 'CP' beams have their vector potentials proportional (respectively) to

$$(0, 0, \psi), \quad (\partial_y\psi, -\partial_x\psi, 0), \quad (\psi, 0, 0) \quad \text{and} \quad (-i\psi, \psi, 0) \tag{26}$$

(The quotation marks indicate that the 'LP' and 'CP' beams are fully linearly and circularly polarized only in the plane wave limit: compare theorems (ii) and (iii) of the previous section.)

What are the universal properties of physically acceptable solutions ψ? We have already seen that seven beam invariants must exist. We also saw that certain textbook properties of electromagnetic beams cannot hold for laterally finite beams. Here we argue that an infinity of zeros of ψ must occur in the focal plane.

The solutions $\psi(\mathbf{r})$ of the Helmholtz equation are, in general, complex functions of position, $\psi = \psi_r + i\psi_i$. The real and imaginary parts ψ_r and ψ_i are (in free space) smooth functions of position. These functions are zero on surfaces S_r and S_i, and where these surfaces meet (on curves C in space) both ψ_r and ψ_i are zero. If we write

$$\psi(\mathbf{r}) = M(\mathbf{r})e^{iP(\mathbf{r})} = \left[\psi_r^2 + \psi_i^2\right]^{\frac{1}{2}} \exp\left(i \arctan \frac{\psi_i}{\psi_r}\right) \tag{27}$$

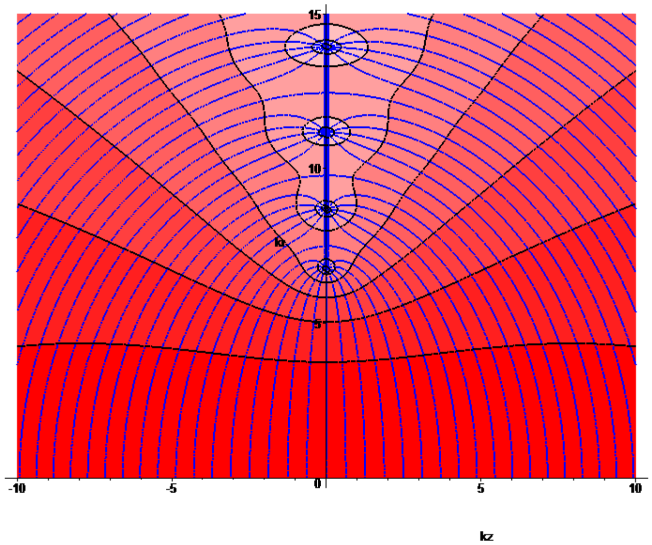

Figure 1: Isophase surfaces (blue) of the $j_0(KR)$ beam, Equation (5), plotted for $Kb = 6$, at intervals of $\pi/6$. (The three-dimensional picture is obtained by rotating about the z-axis.) The surfaces with phase equal to an integer multiple of π converge onto the circles $\rho = [(X/K)^2 + b^2]^{\frac{1}{2}}$, where $\tan X = X$. The other isophase surfaces converge onto the zeros of $j_0(KR)$ in the $z = 0$ plane, namely on the circles $\rho_n = [(n\pi/K)^2 + b^2]^{\frac{1}{2}}$. Surfaces of constant modulus are shown in black.

we see that, on any such curve C, the modulus $M(x)$ is zero, and the phase $P(\mathbf{r})$ is indeterminate. Nye and Berry [13] called these curves wave dislocations; Chapter 5 of Nye's book [14] gives illustrations of such phase singularities.

We give a topological argument for the existence of zeros of ψ in the focal plane, on the assumption that the isophase surfaces intersect the focal plane. At the zeros of ψ the phase can be any real number, but we exclude integer multiples of π, as explained below.

The focal plane is a plane of symmetry for an ideal beam; we can take it to be the $z = 0$ plane, and can also take the phase of ψ to be zero at the origin. Then the isophase surfaces correspond to negative $P(\mathbf{r})$ for $z < 0$ and positive $P(\mathbf{r})$ for $z > 0$. The surfaces $P = -n\pi$ and $P = n\pi$ can meet where ψ is not zero, since the phase difference is an integer (n) multiple of 2π. These isophase surfaces are concave toward the origin, since a physical beam is converging toward the focal region for $z < 0$ and diverging from it for $z > 0$. The other isophase surfaces can only meet on the focal plane if on it there exist curves where ψ is zero. On such curves (circles, in the simplest case) the phase surfaces $P = -\pi/2$ and $P = +\pi/2$ can meet, for example. The surfaces with $0 < |P| < \pi$ meet on the first zero curve, $\pi < |P| < 2\pi$ meet on the next, and so on. Figure 1 illustrates phenomenon, which I conjecture to be universal at the focal plane. Because of the topological nature of the above argument, we expect the zeros to persist even when the beam is perturbed (for example, focused by an imperfect lens or mirror). The focal plane would then be distorted to a nearly-planar surface, and the circles of zeros to approximately circular closed curves, where the perturbed phase surfaces $\pm P$ meet.

One counter-example to the above conjecture (of the universality of rings of zeros in the focal plane) appears to be separable spheroidal beams, for which $\psi(\xi, \eta, \phi) = R(\xi)S(\eta)e^{im\phi}$, with $\rho = b[(\xi^2 + 1)(1 - \eta^2)]^{\frac{1}{2}}$, $z = b\xi\eta$. The focal plane $z = 0$ corresponds to $\xi = 0$ for $\rho \leq b$ and $\eta = 0$ for $\rho \geq b$. Thus if $S(\eta)$ is zero for $\eta = 0$, $\psi = 0$ for $\rho \geq b$ in the focal plane, and the $-P$ and $+P$ isophase surfaces can meet anywhere on the focal plane outside of the central disk $\rho \leq b$. However, such spheroidal wavefunctions have been shown to be non-physical [15].

REFERENCES

1. Jackson, J. D., *Classical Electrodynamics*, 2nd Edition, Wiley, New York, 1975.
2. Deschamps, G. A., "Gaussian beam as a bundle of complex rays," *Electron. Lett.*, Vol. 7, 684–685, 1971.
3. Sheppard, C. J. R. and S. Saghafi, "Beam modes beyond the paraxial approximation: A scalar treatment," *Phys. Rev. A*, Vol. 57, 2971–2979, 1998.

4. Ulanowski, Z. and I. K. Ludlow, “Scalar field of nonparaxial Gaussian beams,” *Opt. Lett.*, Vol. 25, 1792–1794, 2000.
5. Lekner, J., “TM, TE and ‘TEM’ beam modes: Exact solutions and their problems,” *J. Opt. A: Pure Appl. Opt.*, Vol. 3, 407–412, 2001.
6. Lekner, J., “Invariants of three types of generalized Bessel beams,” *J. Opt. A: Pure Appl. Opt.*, Vol. 6, 837–843, 2004.
7. Bateman, H., “The solution of partial differential equations by means of definite integrals,” *Proc. Lond. Math. Soc.*, Vol. 1, 451–458, 1904.
8. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd Edition, Cambridge University Press, 1966.
9. Lekner, J., “Invariants of electromagnetic beams,” *J. Opt. A: Pure Appl. Opt.*, Vol. 6, 204–209, 2004.
10. Barnett, S. M., “Optical angular momentum flux,” *J. Opt. B: Quantum Semiclass.*, Vol. 4, S7–S16, 2002.
11. Lekner, J., “Polarization of tightly focused laser beams,” *J. Opt. A: Pure Appl. Opt.*, Vol. 5, 6–14, 2003.
12. Lekner, J., “Phase and transport velocities in particle and electromagnetic beams,” *J. Opt. A: Pure Appl. Opt.*, Vol. 4, 491–499, 2002.
13. Nye, J. F. and M. V. Berry, “Dislocations in wave trains,” *Proc. Roy. Soc. Lond. A*, Vol. 336, 165–190, 1974.
14. Nye, J. F., *Natural Focusing and Fine Structure of Light*, Inst. Phys. Publishing, Bristol, 1999.
15. Boyack, R. and J. Lekner, “Non-existence of separable spheroidal beams,” *J. Opt.*, Vol. 13, 085701(3), 2011.