Regions of attraction between like-charged conducting spheres

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Two positively charged conducting spheres have been shown to attract at close enough range, unless they have a charge ratio that would result from contact. We give analytical results for the charge ratio at which the cross-over between electrostatic attraction and repulsion occurs, as a function of the sphere separation.
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The purpose of this note is to give a precise criterion for the existence of attraction between like-charged spheres. The criterion is that at a given sphere separation the charge ratio must either exceed a critical value or be smaller than a critical value. Outside of this region, for large enough or small enough values of the charge ratio, the spheres will attract. We give analytical results in the case of conducting spheres of equal radii.

The theory is based on Ref. 1, where attraction of like-charged conducting spheres at small separation was predicted by analytical means. Naturally, this prediction faced healthy scepticism. For example, Griffiths wrote,[2] "Suppose I take two conducting spheres, of equal radius and equal charge. They repel, regardless of the separation distance. Paint a thin layer of insulation over their surfaces, and place them in contact (or, if you like, leave them uninsulated and place them a tiny distance apart). Add an infinitesimal charge (maybe a single electron) to one of them. Now they attract (by your theorem). Are we really to believe that this minute increment in charge switches a force that was large and repulsive to one that is (perhaps small but) attractive?"

The answer given below amounts roughly to the following: if a small charge is added to one of the spheres, the spheres will attract at a separation that is proportional to the square of the deviation of the charge ratio from unity. (A more precise criterion will be given later.) Thus, a charge deviation from unity of one part per thousand requires approach to within one millionth of the radius. As discussed in Ref. 1, surface roughness of the spheres becomes important at very small separations, and is likely to lead to an electrical short. Electron tunneling through a very thin layer of insulator (as in the Griffiths scenario above) is another way in which charges on the spheres can equalize.

The analytical results of Ref. 1 for the capacitance coefficients and for the charge distributions on the two spheres have been confirmed numerically.[3] In addition, generalizations to pairs of non-spherical conductors have been explored.[4,5] In each case the attraction between like-charged conductors, where it exists, can be traced to the redistribution of surface charge on the conductors. Figure 7 of Ref. 1 and Figs. 3 and 4 of Ref. 3 show the charge distribution on spheres of radii a and $2a$ with positive total charges $2Q$ and Q, respectively. The separation between the spheres is $s = a$ (s is the separation distance of the closest points of the two spheres), not close enough for net attraction to occur, but already there is an induced *negative* surface charge on the part of the larger sphere closest to the smaller one.

Figure 1 of this paper shows the equipotentials for this configuration, for which the potential of the smaller sphere is larger by a factor of about 2.17 than that of the larger sphere (zero potential at infinity being assumed). The variable charge density is due to very small displacements (relative to the nuclei) of a very large number of conduction electrons. The displacements can produce locally a negative surface charge on a positively charged sphere.

The electrostatic energy of two conductors a, b at potentials V_a, V_b carrying charges Q_a, Q_b is

$$W = \frac{1}{2}Q_aV_a + \frac{1}{2}Q_bV_b. \tag{1}$$

The charges are related to the potentials via Maxwell's capacitance coefficients (Ref. 6, Sec. 87)

$$Q_a = C_{aa}V_a + C_{ab}V_b, \quad Q_b = C_{ab}V_a + C_{bb}V_b. \tag{2}$$

Inverting these equations gives the potentials in terms of the charges and the capacitance coefficients

$$V_a = \frac{Q_aC_{bb} - Q_bC_{ab}}{C_{aa}C_{bb} - C_{ab}^2}, \quad V_b = \frac{Q_bC_{aa} - Q_aC_{ab}}{C_{aa}C_{bb} - C_{ab}^2}. \tag{3}$$

Thus the potential energy of two conductors (of arbitrary shape) carrying charges Q_a, Q_b is

$$W = \frac{Q_a^2C_{bb} - 2Q_aQ_bC_{ab} + Q_b^2C_{aa}}{2\left(C_{aa}C_{bb} - C_{ab}^2\right)}. \tag{4}$$

When the conductors are spheres of radii a, b, the capacitance coefficients depend only on a, b and the distance between their centers $c = a + b + s$. Maxwell (Ref. 6, Sec. 173) was the first to find the capacitance coefficients for two spheres. They are infinite sums over ratios of hyperbolic functions of integral multiples of the variable U, defined by

$$\cosh U = \frac{c^2 - a^2 - b^2}{2ab} = 1 + \frac{s}{a} + \frac{s}{b} + \frac{s^2}{2ab}. \tag{5}$$

When c is large compared to the sum $a + b$ of their radii, the leading terms of the expansion of the energy in inverse

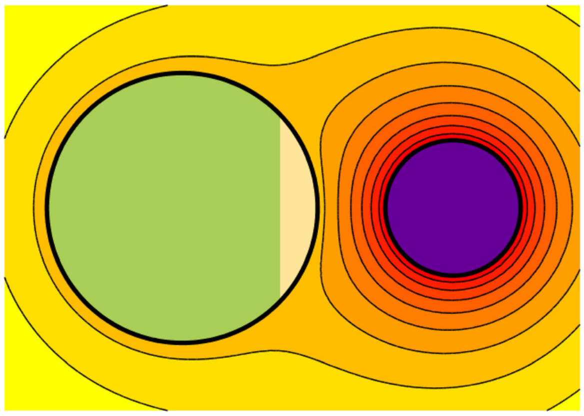

Fig. 1. Equipotential contours of two charged conducting spheres with radii a, $2a$, and charges $2Q$, Q. On each sphere, the shading (color online) indicates the sign and magnitude of the surface charge: on the small sphere there is a large positive charge density (dark shading); on the larger sphere the gray denotes a smaller positive charge density and the lightest shade a negative charge density.

powers of c (which in turn follows from the expansion of the capacitance coefficients in inverse powers of c) are

$$W = \frac{Q_a^2}{2a} + \frac{Q_b^2}{2b} + \frac{Q_a Q_b}{c} - \frac{Q_a^2 b^3 + Q_b^2 a^3}{2c^4} - \frac{Q_a^2 b^5 + Q_b^2 a^5}{2c^6} + \cdots . \tag{6}$$

The first two terms are the self-energies of the two charged spheres, then comes the Coulomb energy, followed by an attractive term that originates in the mutual polarization of the spheres, as explained in Ref. 1. The next term is also attractive. Remarkably, Maxwell[6] carried out an equivalent expansion in reciprocal powers of the sphere separation to order c^{-22}! (Maxwell's calculation is discussed in Ref. 7.)

The force between two widely separated spheres is therefore

$$F = -\partial_c W = \frac{Q_a Q_b}{c^2} - 2\frac{Q_a^2 b^3 + Q_b^2 a^3}{c^5} + \cdots . \tag{7}$$

Assuming Q_a and Q_b have the same sign, to this approximation the force will be attractive (negative) for

$$c^3 < 2\left(\frac{Q_a}{Q_b} b^3 + \frac{Q_b}{Q_a} a^3\right). \tag{8}$$

Attraction between two widely separated like-charged spheres is thus possible only for a charge ratio very different from unity. At a given (large) separation, the required charge ratios are

$$\frac{Q_b}{Q_a} \approx \frac{1}{2}\left(\frac{c}{a}\right)^3 \quad \text{or} \quad \frac{Q_b}{Q_a} \approx 2\left(\frac{b}{c}\right)^3. \tag{9}$$

The close-approach situation is more interesting. The capacitance coefficients have terms logarithmic in the sphere separation, and lead to the prediction that attraction will occur when the spheres are close enough to each other, unless they have the charge ratio that they would attain on contact. The question "how close is close enough for attraction?" is answered in general as follows.

To find the electrostatic force between the spheres, we differentiate the energy in Eq. (4) with respect to the sphere separation. We find that $F = -\partial_c W = -\partial_s W$ is zero when the charge ratio $q = Q_b/Q_a$ satisfies the quadratic

$$A_0 + A_1 q + A_2 q^2 = 0, \tag{10}$$

where

$$A_0 = 2C_{bb}C_{ab}D_{ab} - C_{bb}^2 D_{aa} - C_{ab}^2 D_{bb}, \tag{11}$$

$$A_1 = 2C_{ab}(C_{aa}D_{bb} + C_{bb}D_{aa}) - 2(C_{aa}C_{bb} + C_{ab}^2)D_{ab}, \tag{12}$$

and

$$A_2 = 2C_{aa}C_{ab}D_{ab} - C_{aa}^2 D_{bb} - C_{ab}^2 D_{aa}. \tag{13}$$

The coefficients D are derivatives of the capacitance coefficients with respect to separation, $D_{ab} = \partial_c C_{ab} = \partial_s C_{ab}$, etc. Equivalently, we can replace derivatives with respect to c or s by derivatives with respect to the dimensionless coordinate U (the sum of the bispherical coordinates of the two spheres) defined in Eq. (5). (U increases monotonically with c or s, so the zero-force quadratic is the same in either case.) Equation (10) is simultaneously a transcendental equation for the zero-force sphere separation in terms of the charge ratio, but it is simpler to treat it as a quadratic in $q = Q_b/Q_a$.

The large-separation capacitance coefficients of Ref. 1 inserted into Eq. (10) lead to Eqs. (8) and (9). The discriminant of the quadratic (10) is

$$A_1^2 - 4A_0A_2 = 4(C_{aa}C_{bb} - C_{ab}^2)^2(D_{ab}^2 - D_{aa}D_{bb}). \tag{14}$$

For large separation, we find that this is positive, with leading term $4a^4b^4/c^4$, decreasing with sphere separation. At close approach, the discriminant is positive and increasing with separation. Thus the quadratic for q has two real roots in each limiting case. That two real charge-ratio solutions exist at any separation would follow from an analytic proof of the inequality $D_{ab}^2 - D_{aa}D_{bb} > 0$, which seems difficult. A physical argument for the existence of cross-over between repulsion and attraction at any separation applicable to spheres of equal radii, based on a suggestion of Griffiths, goes as follows. For any value of the separation distance, if one of the charges is zero the force is obviously attractive, and for equal charges it is by symmetry repulsive, so there must be some cross-over charge ratio where it goes from attractive to repulsive.

We now look in more detail at the *close-approach behaviour*. Reference 1 gives the capacitance coefficients to order U^2 for general a, b. The equal-radii coefficients are simpler

$$C_{aa} = \frac{a}{2}\left\{\ln\frac{2}{U} + \gamma + 2\ln 2 + \frac{U^2}{24} \times \left[\ln\left(\frac{2}{U}\right) + \gamma + 2\ln 2 - \frac{1}{6}\right]\right\} = C_{bb}, \tag{15}$$

$$C_{ab} = -\frac{a}{2}\left\{\ln\left(\frac{2}{U}\right) + \gamma + \frac{U^2}{24}\left[\ln\left(\frac{2}{U}\right) + \gamma + \frac{1}{3}\right]\right\}, \tag{16}$$

($\gamma \approx 0.5772$ is Euler's constant). When $a = b$ there is extra symmetry in the quadratic (10) for the charge ratio

$q = Q_b/Q_a$: $A_0 = A_2$, so the quadratic is unchanged on the replacement of q by q^{-1}, as expected. Equation (10) now gives the cross-over charge ratio as

$$q + q^{-1} = -\frac{A_0}{A_1} = 2 + O(U^2). \tag{17}$$

The term of order U^2 is positive, so the quadratic has real solutions, as noted above. The solutions are also positive, and reciprocals of each other: $q_\pm = (1+\delta)^{\pm 1}$, where δ^2 is equal to the $O(U^2)$ term in Eq. (17), which follows from Eqs. (10), (15), and (16):

$$\delta^2 \approx \frac{4\ln 2 - 1}{12}\left(\ln\frac{4}{U} + \gamma\right)^2 U^2. \tag{18}$$

For spheres of equal radii, $U \approx 2(s/a)^{1/2}$ at close approach, and the cross-over charge ratios differ from unity by

$$\delta \approx \left(\frac{4\ln 2 - 1}{12}\right)^{1/2}\left[\ln\left(\frac{4a}{s}\right) + 2\gamma\right]\left(\frac{s}{a}\right)^{1/2}. \tag{19}$$

Figure 2 shows the degree of repulsion or attraction between two equal spheres, as a function of their separation and of the charge ratio $q = Q_b/Q_a$. The contours give the ratio of the force between the spheres to the Coulomb force between point charges Q_a, Q_b separated by the center-to-center distance c between the spheres, namely $F_C = Q_a Q_b/c^2$. The thick contours give the charge ratio at which the force is zero, and thus bound the region of repulsion.

From Eq. (19) we see that the dominant variation in the charge ratio for which the force is zero is (at close approach) determined by the square root of separation to radius ratio. Conversely, the separation at which cross-over to attraction occurs varies as the square of the deviation of the charge ratio from unity, $s/a \sim \delta^2 = (Q_b/Q_a - 1)^2$.

The theoretical F/F_C contours in Fig. 2 all meet at the singular contact point (0, 1) in the (*separation/sphere diameter, charge ratio*) plane. The neighbourhood of the contact point is interesting. We shall discuss the equal-sphere case; for the general case see Sec. 4 of Ref. 1. Once the spheres touch and the charges are equalized [$Q = (Q_a + Q_b)/2$ is the charge on each sphere after contact], the force is repulsive and given by the Kelvin[7,8] expression at contact, namely, by

$$F_0 = \frac{4\ln 2 - 1}{6(\ln 2)^2}\frac{Q^2}{(2a)^2} \approx 0.6149\frac{Q^2}{(2a)^2}. \tag{20}$$

The force remains repulsive at all distances, by symmetry. It is almost constant at small separations.[1] Before the charges are equalized the leading close-approach term of the force is attractive. It increases with decreasing separation and is proportional to the charge difference squared[1]

$$F = -\frac{(Q_a - Q_b)^2}{2as\left[\ln\left(\frac{4a}{s}\right) + 2\gamma\right]} + O(1). \tag{21}$$

If the charge difference is very small, the next (repulsive) term can dominate at finite separation, but the repulsion does not exceed the Kelvin contact force. So, near the contact point the force can be attractive or repulsive, depending on the charge difference and the separation. The repulsion is bounded by the Kelvin contact force; the attraction is in theory unbounded but in practice eventually converted into repulsion by an electrical short at close approach.

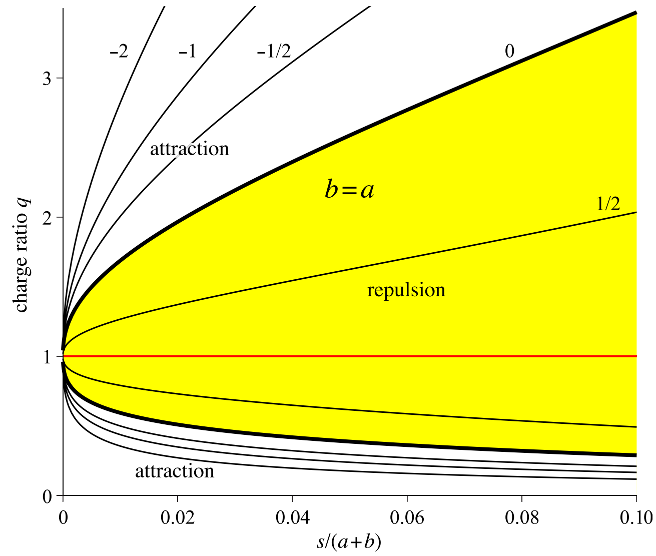

Fig. 2. Regions of attraction and repulsion in the (separation, charge ratio) plane, plotted for spheres of equal radii. When the charges on the two spheres are equal (charge ratio equal 1), there is repulsion at all separations. For unequal charges, two regions of attraction exist, bounded by the reciprocal curves derived in the text (thick solid curves). The contours correspond to the values of the ratio of the force to the point-charge Coulomb force $F_C = Q_a Q_b/c^2$. The values are $\frac{1}{2}$ (in the repulsion region), 0 on the thick curves, and $-\frac{1}{2}, -1, -2$ in the regions of attraction.

The same square root variation of the bounding charge ratio curves persists in the general case of unequal radii. The charge ratio that is attained on contact is, from Eqs. (4.1) and (4.2) of Ref. 1,

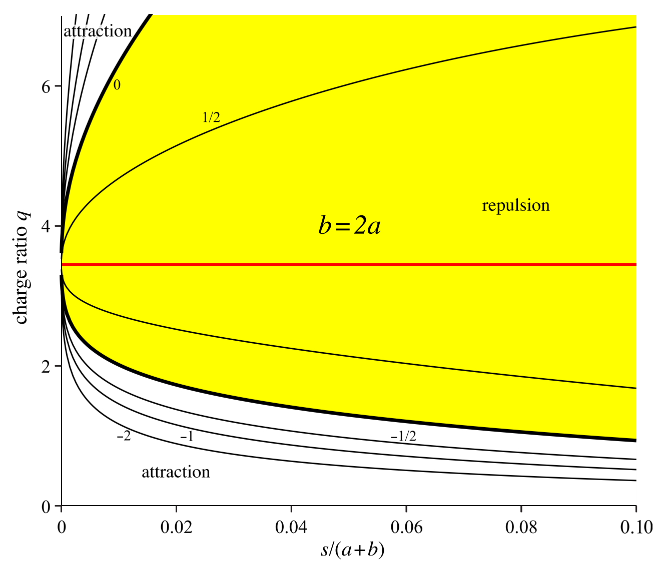

Fig. 3. Regions of attraction and repulsion in the (separation, charge ratio) plane, plotted for spheres with unequal radii, $b = 2a$. When the charges on the two spheres are in the contact ratio given in Eq. (19), there is repulsion at all separations. For charges not in this ratio, two regions of attraction exist, bounded by the thick solid curves. The contours correspond to the values of the ratio of the force to the point-charge Coulomb force $F_C = Q_a Q_b/c^2$. As in Fig. 2, the values are $\frac{1}{2}$ (in the repulsion region), 0 on the thick curves, and $-\frac{1}{2}, -1, -2$ in the regions of attraction.

$$\frac{Q_b}{Q_a} = \frac{\gamma + \psi\left(\frac{a}{a+b}\right)}{\gamma + \psi\left(\frac{b}{a+b}\right)} \approx \left(\frac{b}{a}\right)^2 \left(\frac{\pi^2}{6}\right)^{\frac{a-b}{a+b}},$$

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z). \quad (22)$$

For example, when $b = 2a$ the contact charge ratio (for which there is repulsion at all separations) is

$$\frac{Q_b}{Q_a} = \frac{\gamma + \psi(1/3)}{\gamma + \psi(2/3)} = \frac{9\ln 3 + \pi\sqrt{3}}{9\ln 3 - \pi\sqrt{3}} \approx 3.4477. \quad (23)$$

[The value in Eq. (23) is 1.7% larger than is given by the approximate expression in Eq. (22).]

For unequal radii, the bounding curves of the repulsion region are no longer reciprocal, as they are in the equal radii case. They meet at zero separation at the charge ratio given in Eq. (22). Figure 3 shows the contours of the force ratio F/F_C and the regions of repulsion and attraction in the case $b = 2a$.

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Hewlett-Packard HP35 Calculator

This is a rather recent antique. I spent the academic year 1972-73 at an overseas university where the students multiplied and divided using logarithm tables. When I returned to Kenyon and became chair of the physics department, I invested a large fraction of the apparatus budget in two HP35 calculators, which were held in security cradles with steel cables tying them down to the laboratory benches. Very few students had their own calculators, and we made sure that they had their names engraved on them. Now the students arrive with graphing calculators from their high school mathematics courses, and actually use them relatively little. This calculator has been retired to the Greenslade Museum. (Notes and picture by Thomas B. Greenslade, Jr., Kenyon College)