To obtain the real physical polarizations, one multiplies the right-hand side of each equation by $\exp(-i\Omega t)$ and takes the real part.

[13]Upon emerging from the sample, the complex waveform is $\mathbf{E}_t = \mathbf{R}\exp(in_-\Omega l/c) + \hat{\mathbf{L}}\exp(in_+\Omega l/c)$. Factoring out $\exp[i(n_+ + n_-)\Omega l/2c]$ and substituting relations (A) of Ref. 12 leads to Eqs. (20a) and (20b).

[14]At normal incidence on an isotropic medium, there is no distinction between transverse electric and transverse magnetic waves. Two orthogonal states of linear polarization are therefore reflected with a Fresnel amplitude of the same magnitude and phase. Thus the reversal of the propagation vector interchanges the helicity of RCP and LCP waves. See, for example, Ref. 10, pp. 94–96.

[15]The Faraday effect is a magnetically induced optical activity. It differs from natural optical activity in that the rotation of the plane of polarization occurs with respect to the static magnetic field and not with respect to the propagation vector. Reverse passage of a light beam through a naturally optically active medium leads to an optical rotation in the opposite sense as that of the initial passage. See Ref. 6, pp. 222–227.

[16]The Cotton–Mouton effect is the magnetic analog of the Kerr effect, or electric field-induced birefringence, frequently used in electro-optics as a means of switching light. For both effects, see Ref. 6, pp. 227–230.

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The phase relation between reflected and transmitted waves, and some consequences

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The relative phase of waves reflected from and transmitted through a symmetric stratification is (under certain constraints) known to be an odd multiple of $\pi/2$. The phase difference is shown to change by π when the reflectivity passes through zero. Continuity arguments indicate that a surprisingly large proportion of symmetric stratifications share the property of zero reflectance at certain angles. The results apply to acoustic and particle waves, as well as to electromagnetic waves of both polarizations.

I. INTRODUCTION

Degiorgio,[1] Zeilinger,[2] and Lai *et al.*[3] have discussed the phase relationship of the reflected and transmitted optical fields of a semireflecting mirror. Let r and t denote the reflection and transmission amplitudes (these complex numbers are defined below), and $ph(A)$ denote the real quantity δ in $A = |A|\exp(i\delta)$. Degiorgio asserted that $ph(r/t) = \pi/2$ for a lossless beam splitter. References 2 and 3 noted that in fact it had to be a *symmetric* lossless beam splitter for this phase relationship to hold. Here, we give two general proofs that $ph(r/t)$ is an odd multiple of $\pi/2$ under certain conditions, show that this relative phase switches by π as the reflectivity passes through zero (either angle of incidence or reflector parameters being the variables), and draw some unexpected conclusions.

II. THE PHASES OF THE REFLECTED AND TRANSMITTED WAVES

We first note that it is only under special conditions that there is a fixed relationship between the phases of the reflected and transmitted waves. Consider a wave field

$$\Psi(z,x,t) = \psi(z)\exp[i(Kx - \omega t)], \tag{1}$$

which represents a plane wave propagating in the z and x directions, and incident on a planar stratification (usually localized within some range of z, say, z_1 to z_2). For electromagnetic, acoustic, and particle waves, the function $\psi(z)$ satisfies an ordinary linear second-order differential equation. For example, for the electromagnetic s(TE) wave, or for particle (Schrödinger) waves, ψ satisfies[4]

$$\frac{d^2\psi}{dz^2} + q^2\psi = 0,$$

where

$$q^2(z) = \epsilon(z)(\omega^2/c^2) - K^2,$$

or

$$(2M/\hbar^2)[E - V(z)] - K^2. \tag{2}$$

Here, $\epsilon(z)$ is the dielectric function, c is the speed of light, ω is the angular frequency, E and V are the total and potential energies of the particle of mass M. The wavenumber K is the component of the wave vector along the x direction and is a constant of the motion. If $\epsilon(z)$ and $V(z)$ eventually take constant values as $z \to \pm\infty$, $q(z)$ tends to the limiting values q_1 and q_2, and, for waves incident from negative z, ψ has the limiting forms

$$\exp(iq_1 z) + r\exp(-iq_1 z) \leftarrow \psi \rightarrow t\exp(iq_2 z). \tag{3}$$

Equation (3) defines the reflection and transmission amplitudes r and t. [For the electromagnetic p(TM) wave, and for acoustic waves, the differential equations are a little more complicated,[4] but the form of (3) is the same.]

It is clear from (3) that the phases of r and t in general depend on the choice of origin. For example, if we shift the origin of z, setting $z \to z - a$, the new reflection and transmission amplitudes [defined as the coefficients of $\exp(-iq_1 z)$ and of $\exp(iq_2 z)$ when the incident wave is $\exp(iq_1 z)$] are given by

$$r \to r\exp(2iq_1 a), \quad \text{and} \quad t \to t\exp[i(q_1 - q_2)a]. \tag{4}$$

We see that r always depends on the choice of origin, while t is independent of it only when $q_1 = q_2$ (the media on either side of the reflecting object have the same ϵ or V or acoustic characteristics). To see these results physically, imagine a fixed loudspeaker radiating sound and a partial reflector being moved toward or away from the source. As the reflector moves, the phase relationship between the incident and reflected waves changes, but the phase relation between the incident and transmitted waves will not change if the media on either side of the reflector are acoustically identical.

An alternative definition of reflection and transmission amplitudes, suggested by one of the referees and based on the point of view taken in Ref. 3, is via

$$\exp[iq_1(z - z_1)] + r_1\exp[-iq_1(z - z_1)]$$
$$\leftarrow \psi \rightarrow t_2\exp[iq_2(z - z_2)]. \tag{3'}$$

This defines r_1 as the reflected wave at the point z_1, and t_2 as the transmitted wave at the point z_2 {the common phase factor $\exp[i(Kx - \omega t)]$ being suppressed}. Comparison with (3) shows that

$$r = \exp(2iq_1 z_1) r_1, \quad t = \exp[i(q_1 z_1 - q_2 z_2)] t_2.$$

Note that for reflectors immersed in a single medium and with z_1 and z_2 chosen so that $z_1 + z_2 = 0$, r and t are, respectively, equal to r_1 and t_2 multiplied by the same phase factor $\exp(2iq_0 z_1)$.

An advantage of (3') is that r_1 and t_2 are independent of the choice of origin of z, and so is the phase of r_1/t_2, in all cases. A disadvantage is that while a natural choice for z_1 and z_2 is at the boundaries of the reflector, such boundaries are not well defined in many cases of interest. [Two examples are a diffuse liquid–vapor interface, and an ionospheric layer, e.g., as approximated by (18).]

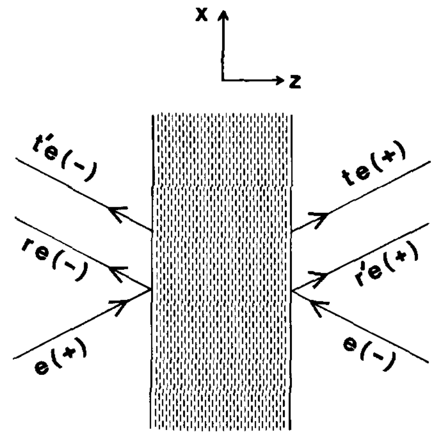

Fig. 1. Symmetrically incident plane waves on the left and right of an arbitrary stratification. The reflection amplitudes r,t and r',t' refer to waves incident from the left and from the right, respectively. The factors $e(\pm)$ stand for $\exp i(\pm q_0 z + Kx - \omega t)$. Note that the relative phase of the two incident waves is not arbitrary: It is zero in the $z = 0$ plane if the stratification is symmetric about $z = 0$.

From now on we will consider the identical media case ($\epsilon_1 = \epsilon_2$ or $V_1 = V_2$, etc.). Let q_0 be the common value of q_1 and q_2, and consider the Degiorgio beam-splitter configuration shown in Fig. 1.

For the configuration shown, if the input intensity is 2, the output intensity is

$$|r + t'|^2 + |r' + t|^2 = |r|^2 + |t|^2 + |r'|^2 + |t'|^2 + 2\,\mathrm{Re}(r^*t') + 2\,\mathrm{Re}(t^*r'). \tag{5}$$

If there is no absorption,

$$|r|^2 + |t|^2 = 1 = |r'| + |t'|^2. \tag{6}$$

[This follows from the more general conservation law $q_1(1 - |r|^2) = q_2|t|^2$; see Ref. 4, Sec. 2-1.] Thus

$$\mathrm{Re}(r^*t') + \mathrm{Re}(t^*r') = 0. \tag{7}$$

Now, $t' = t$ [a special case of Eq. (2.14) of Ref. 4], and $r' = -(t/t^*)r^*$ [Eq. (2.18) of Ref. 4]. Thus (7) reduces to the identity $\mathrm{Re}(r^*t) + \mathrm{Re}(-tr^*) = 0$, and no new result is contained in the above.

There *is* a new consequence in the *symmetric stratification* case, where the output intensity (in the absence of absorption) must be unity on each side:

$$|r + t'|^2 = 1 = |r' + t|^2. \tag{8}$$

These equations, together with the above general relations, imply that

$$\mathrm{Re}(rt^*) = 0, \quad ph(r/t) = \text{odd multiple of } \pi/2. \tag{9}$$

We will give another proof of (9) in terms of wavefunction $\psi(z)$ within the layer. We assume the region of variable $\epsilon(z)$ or $V(z)$ to be contained within (z_1, z_2). Since ψ satisfies the second-order differential equation (2), it may be written as a superposition of two linearly independent solutions of (2), say $F(z)$ and $G(z)$. Thus

$$\psi(z) = \begin{cases} \exp(iq_0 z) + r\exp(-iq_0 z), & z < z_1, \\ \alpha F(z) + \beta G(z), & z_1 \leqslant z \leqslant z_2, \\ t\exp(iq_0 z), & z > z_2. \end{cases} \tag{10}$$

Continuity of ψ and $d\psi/dz$ at z_1 and z_2 gives four equations in the four unknowns r, α, β, t. We find

$$r = \exp(2iq_0 z_1)\,\frac{q_0^2(F_1G_2 - G_1F_2) + iq_0(F_1G_2' - G_1F_2') + iq_0(F_1'G_2 - G_1'F_2) - (F_1'G_2 - G_1'F_2')}{q_0^2(F_1G_2 - G_1F_2) + iq_0(F_1G_2' - G_1F_2') - iq_0(F_1'G_2 - G_1'F_2) + (F_1'G_2 - G_1'F_2')},$$

$$t = \exp[iq_0(z_1 - z_2)]\,\frac{2iq_0(F_2G_2' - G_2F_2')}{q_0^2(F_1G_2 - G_1F_2) + iq_0(F_1G_2' - G_1F_2') - iq_0(F_1'G_2 - G_1'F_2) + (F_1'G_2 - G_1'F_2')}. \quad (11)$$

[These equations are the $q_1 = q_0 = q_2$ special case of the general formulas (2.25) and (2.26) of Ref. 4.] In the above, F_1 stands for $F(z_1)$, F_1' for dF/dz evaluated at z_1, etc.

If we chose $z_1 = -z_2$ (the boundaries of the reflecting stratification symmetrically placed about the origin), the phase factors premultiplying the r and t expressions become the same. If, further, the stratification is symmetric about the plane $z = 0$, then $q^2(z)$ is symmetric about $z = 0$, and the functions F and G can be chosen to be even and odd, respectively. (In the language of quantum mechanics, parity is a good quantum number.) Thus, in this special case, $F(-z) = F(z)$, $G(-z) = -G(z)$, $F'(-z) = -F'(z)$, $G'(-z) = G'(z)$, and

$$F(-z)G'(z) - G(-z)F'(z) + F'(-z)G(z) - G'(-z)F(z) = 0, \quad (12)$$

for all z within the symmetric and symmetrically placed stratification. In particular, the combination $F_1G_2' - G_1F_2' + F_1'G_2 - G_1'F_2$ appearing in the numerator of r is zero, so that the numerator of r is a real quantity. [The functions F and G can be chosen to be real, since $q^2(z)$ is real in the absence of absorption.] Since the numerator of t is imaginary, and r and t share a common denominator, it follows that r/t is an imaginary quantity, which proves (9).

III. PHASE SWITCHING AND ZERO REFLECTIVITY

We have seen that symmetric stratifications have r/t imaginary, when the origin is taken in the plane of symmetry. Thus the phase of r/t changes by π as the reflectivity passes through zero (for variable angle of incidence, or the thickness of the stratification, or some characterization of the stratification profile). The switch is sudden and exactly π in the case of zero absorption, but becomes continuous and only approximately π when there is absorption. Figure 2 illustrates $ph(r/t)$ for the electromagnetic s wave and a uniform layer of thickness Δz. From Ref. 4, Eqs. (2.58) and (2.59), or Ref. 5, Eqs. (1.57) and (1.58),

$$r = \exp(2iq_0 z_1) \times \{r_0[1 - \exp(2iq\,\Delta z)]/[1 - r_0^2 \exp(2iq\,\Delta z)]\}, \quad (13)$$

$$t = \exp(-iq_0\,\Delta z) \times \{(1 - r_0^2)\exp(iq\,\Delta z)/[1 - r_0^2 \exp(2iq\,\Delta z)]\}, \quad (14)$$

where the layer extends from z_1 to $z_1 + \Delta z$. Here r_0 is the Fresnel reflection amplitude

$$r_0 = (q_0 - q)/(q_0 + q). \quad (15)$$

When $z = 0$ is the center of the uniform layer, $z_1 = -\Delta z/2$ and

$$r/t = -[r_0/(1 - r_0^2)]2i \sin q\,\Delta z. \quad (16)$$

In the absence of absorption this is imaginary, and the reflectivity is zero when $q\,\Delta z$ is a multiple of π. (The effective wavelength for the motion in the z direction is $\lambda = 2\pi/q$, and zero reflection occurs when Δz is a multiple of $\lambda/2$; this corresponds to destructive interference of the waves reflected at either side of the dielectric plate.) The zero reflection occurs at angles θ_0 and thicknesses Δz such that

$$\epsilon_0 \sin^2\theta_0 = \epsilon - [m\pi/(\omega\,\Delta z/c)]^2, \quad m = \text{integer} \quad (17)$$

(see Ref. 4, p. 46). When there is absorption in the layer, q and r_0 become complex (Ref. 4, Sec. 8-4), r/t is no longer purely imaginary, and the phase of r/t is a continuous function of the variables. Figure 2 shows $ph(r/t)$ for a nonabsorbing and an absorbing layer, as a function of the angle of incidence.

The reflectivity of the electromagnetic p wave from a uniform layer has the same zeros as that of the s wave and, in addition, is zero at the Brewster angle $\theta_B = \arctan(\epsilon/\epsilon_0)^{1/2}$. This is not a destructive interference effect, being due to the equality of the effective wavenumbers q_0/ϵ_0 and q/ϵ (see Ref. 4, Sec. 1-2). Nor are reflectivity zeros restricted to profiles with discontinuous ϵ: The profile

$$\epsilon = \epsilon_0 + \Delta\epsilon\, \text{sech}^2(z/a) \quad (18)$$

has s reflectivity zero when [Ref. 4, Eq. (4.35) and Fig. 6-11]

$$\Delta\epsilon(\omega a/c)^2 = m(m+1) \quad (m = \text{integer}). \quad (19)$$

In all cases, the phase of r/t switches by π when r passes through zero.

We now use a continuity argument to show that *almost all partial reflectors with symmetric profiles* (in ϵ or V) *which are close in parameter space to a profile* [such as the uniform layer or $\text{sech}^2(z/a)$] *which has reflectivity zeros, will also have reflectivity zeros.*

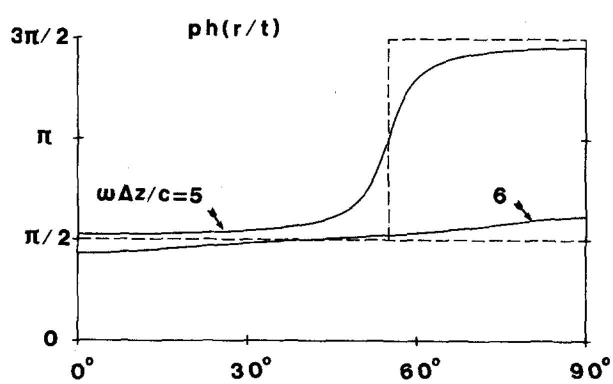

Fig. 2. $ph(r/t)$ vs the angle of incidence, for a layer of dielectric in air with refractive index $\sqrt{\epsilon} = 1.5 + in_i, n_i = 0$ (dashed lines), and 0.02 (solid curves). The values of $(\omega/c)\Delta z$ used are 5 and 6, as indicated. For $(\omega/c)\Delta z = 5$, there is a reflectivity zero at $\theta_0 \approx 55°$ when $n_i = 0$ [from Eq. (17) with $m = 2$].

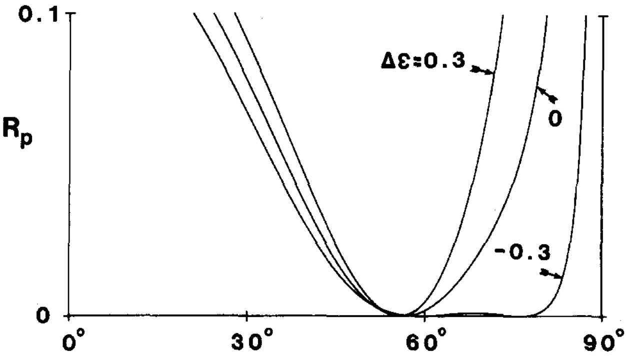

Fig. 3. Reflectivity of the p wave for the symmetric profile given by Eq. (21), for $\omega\,\Delta z/c = 3$, $\epsilon_u = (3/2)^2$, and $\Delta\epsilon = 0$, ± 0.3. (For $\Delta\epsilon = \pm 0.3$ the refractive index at the center of the layer is 1.5 ± 0.1.) The uniform layer has zero reflectivity at $\theta_B = \arctan(3/2) \approx 56.3°$. The $\Delta\epsilon = +0.3$ layer has one reflectivity zero (at 55.8°), while the $\Delta\epsilon = -0.3$ layer has two reflectivity zeros (at 58.9° and 76.3°).

Consider such a profile with similar characteristics to a reference profile that has a reflectivity zero (say, at angle of incidence $= \theta_0$). Somewhat below this angle of incidence, we expect by continuity that the phase of r/t would be the same (say, $\pi/2$) as that of the reference profile, and somewhat above θ_0 to be the same again as that of the reference profile (say, $3\pi/2$). In between, it must change discontinuously. But the only way the phase of r/t can switch is by r passing through zero (where the phase of r is undefined), which proves the assertion.

Note that this argument does not imply that all symmetric profiles will have a reflectivity zero at some angle of incidence. For example, the uniform layer with $(\omega/c)\Delta z = 6$ (shown in Fig. 2) does not—it is not "close enough" in parameter space to the $(\omega/c)\Delta z = 5$ layer. We see that when the reference profile is near to losing its reflectivity zero, even a small change will produce a profile without a reflectivity zero. A case in point is the uniform layer, for which only thicknesses Δz satisfying

$$\frac{m\pi}{\sqrt{\epsilon}} < \frac{\omega}{c}\,\Delta z < \frac{m\pi}{\sqrt{\epsilon - \epsilon_0}} \tag{20}$$

for some integer m will have reflectivity zeros. For $\epsilon_0 = 1$ and $\epsilon = \frac{9}{4}$ (as in Fig. 2), there is a gap between $\omega\,\Delta z/c \approx 5.62$, the upper limit for $m = 2$, and $\omega\,\Delta z/c \approx 6.28$, the lower limit for $m = 3$.

Zero reflectance of the p wave from symmetric profiles is more robust, since the uniform layer has zero p reflectance at $\theta_B = \arctan(\epsilon/\epsilon_0)^{1/2}$, for *any* thickness/wavelength ratio. Figure 3 shows the p reflectance[6] of the symmetric profile

$$\epsilon(z) = \begin{cases} \epsilon_0, & |z| > \Delta z/2, \\ \epsilon_u + \Delta\epsilon[1 - (2z/\Delta z)^2], & |z| < \Delta z/2. \end{cases} \tag{21}$$

For zero $\Delta\epsilon$, this has zero reflectance at $\theta_B = \arctan(\epsilon_u/\epsilon_0)^{1/2}$, for all values of the dimensionless parameter $\omega\,\Delta z/c$. We see that a reflectance zero persists when $\Delta\epsilon \neq 0$, and that in some cases there can be multiple zeros. In the cases shown, one of the profiles has two zeros, and a consequent range of angles (about 20°) in which the reflectivity is very small (less than 10^{-3}). This range of very small reflectivity is about ten times larger than those of the other two profiles shown.

IV. DISCUSSION

We have seen that symmetric nonabsorbing stratifications are likely to show the phenomenon of perfect transmission at some angle or angles. This was deduced from the switching of the phase of the r/t ratio. It is interesting that of the two proofs of the central result (9), the first sheds no light on this property, while the second explicitly demonstrates that for symmetric profiles r is of the form $\exp(2iq_0z_1)N/D$, where N is *real*. Thus the real and imaginary parts of r are automatically zero together when N is zero in contrast to the general case in which a reflectivity zero requires that $\mathrm{Re}(r)$ and $\mathrm{Im}(r)$ be coincidentally zero. For symmetric profiles, it suffices that the real quantity

$$q_0^2(F_1G_2 - G_1F_2) - (F_1'G_2' - G_1'F_2')$$

change sign between normal and grazing incidence for perfect transmission to occur. Symmetric profiles are thus seen to be special and the best candidates for reflection polarizers. The occurrence of multiple zeros can enlarge the range of angles over which the reflectivity is very small. Since the p-wave zero reflectivity is more robust than the s-wave zero reflectivity, p nonreflecting symmetric stratifications are likely to be the more useful reflection polarizers.

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