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TECHNICAL NOTE

Isogyre formation by isotropic refracting bodies

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Summary

A simple derivation is given of a formula for the intensity pattern produced by a diffusely illuminated lens-like body held between a polarizer and an analyser. The formula generalizes the results of Charman and of Pierscionek and Chan for a dielectric sphere between crossed polaroids. A dark cross is universal for all lens-like bodies whose axis of symmetry is normal to the polarizer and analyser. Intensity contours are plotted for ellipsoidal lenses.

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Isogyres are the intensity and colour patterns observed when objects are placed between a polarizer and analyser (usually crossed), and illuminated by light which enters through the polarizer. For uniaxial crystal plates these arrangements produce patterns such as the Maltese cross; biaxial crystal plates produce more complicated patterns (see for example Born and Wolf[1], section 14.4.4 and 14.4.5). Brewster[2,3] noticed the formation of isogyres when the eye lenses of several species were placed between crossed polarizers and ascribed birefringence to the eye on this basis. Pierscionek[4] noted that isotropic dielectric spheres placed between crossed polarizers give isogyric patterns, and gave a qualitative explanation of this in terms of refraction of the light through the spheres. Charman[5] and Piersionek and Chan[6] produced formulae for the intensity pattern produced by a sphere between crossed polarizers. The Pierscionek–Chan intensity simulation based on their formula was similar to Pierscionek's photographs[4], but showed significant differences. Charman's isoluminance contours agree well with his and Pierscionek's photogaphs.

In this note these results are generalized to arbitrary lens-like bodies with homogeneous index. Calculated intensity contours are shown for diffusely illuminated ellipsoidal lenses between crossed polaroids.

Isotropic lens between polarizer and analyser

We shall consider the intensity pattern produced by a lens (with rotational symmetry about its axis) between a polarizer and an analyser, when diffusely illuminated, and viewed from a relatively large distance. We then need to consider only exit rays which are nearly parallel to the axis. The polarizer produces linearly polarized light, which is refracted by the lens, as shown in *Figure 1*.

Let the easy axis of the polarizer be tilted at angle ϕ to the plane of the figure. Then if E_0 is the magnitude of the electric field vibrations in the incident wave and the polarizer is perfect, the p and s polarizations shown in the figure, before the light strikes the lens, have electric field magnitudes

$$E_p = E_0\cos\phi \quad , \quad E_s = E_0\sin\phi \tag{1}$$

The p and s polarizations (relative to the lens, and for rays in the plane of the figure) do not mix if the lens is isotropic, but are transmitted to a different degree. Inside the lens

$$E_p' = t_pE_p \quad , \quad E_s' = t_sE_s \tag{2}$$

and beyond the lens the transmitted light has electric field magnitudes

$$E_p'' = t_p'E_p' \quad , \quad E_s'' = t_s'E_s' \tag{3}$$

where t_p and t_s are the Fresnel transmission amplitudes at the entry point of the ray:

$$t_p = \frac{2\cos\Theta_1\sin\Theta_2}{\sin(\Theta_1+\Theta_2)\cos(\Theta_1-\Theta_2)} \quad , \quad t_s = \frac{2\cos\Theta_1\sin\Theta_2}{\sin(\Theta_1+\Theta_2)} \tag{4}$$

(see, for example, Ref.1. Equation (20a) of Section 1.5.2. or eqns (14) and (32) of Ref.7). The transmission amplitudes

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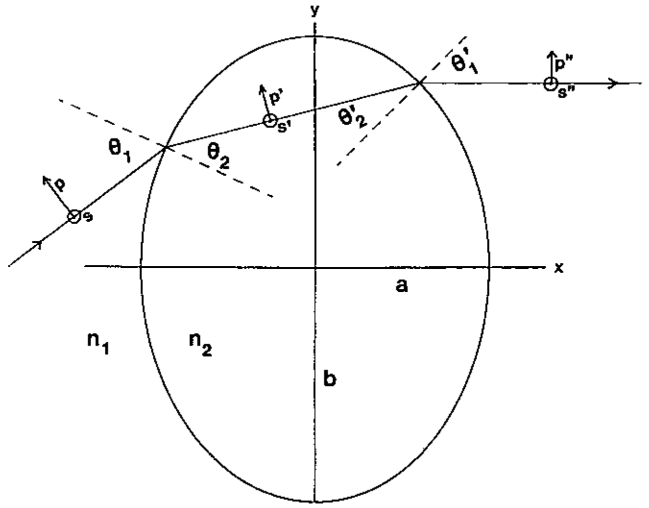

Figure 1. A ray path through an isotropic and homogeneous lens. The light is a mix of p and s polarizations (p has the electric vector in the plane of incidence, s has the electric vector perpendicular to the plane of incidence), as indicated. There is some reflection (not shown) at each refraction. The lens has refractive index n_2, and lies in a medium of index n_1. The diagram is drawn for an ellipsoidal lens with a/b = ¾ and $n_2/n_1 = 1.4$.

at the exit point from the lens are obtained by the substitutions $\Theta_1 \to \Theta_2'$, $\Theta_2 \to \Theta_1'$

$$t_p' = \frac{2\cos\Theta_2' \sin\Theta_1'}{\sin(\Theta_1' + \Theta_2')\cos(\Theta_1' - \Theta_2')}, \quad t_s' = \frac{2\cos\Theta_2' \sin\Theta_1'}{\sin(\Theta_1' + \Theta_2')} \tag{5}$$

The electric field amplitudes of the exit ray are

$$E_p' = t_p t_p' E_p \quad , \quad E_s'' = t_s t_s' E_s \tag{6}$$

Note that the total deviation of the ray by the two refractions is $\Psi = \Theta_1 - \Theta_2 + \Theta_1' - \Theta_2'$. Let the axis of the analyser be tilted at $\phi + \chi$ to the plane of the figure (χ is thus the angle between the easy axes of the polarizer and analyser). Then neglecting the effect of the tilt of the p-polarized component relative to the symmetry axis of the system, the electric field along the analyser axis is

$$\begin{aligned} E_a &= E_p'' \cos(\phi + \chi) + E_s'' \sin(\phi + \chi) \\ &= E_0 \{ t_p t_p' \cos\phi \ \cos(\phi + \chi) + t_s t_s' \sin\phi \ \sin(\phi + \chi) \} \end{aligned} \tag{7}$$

When polarizer and analyser are crossed ($\chi = \pm 90°$), E_a becomes

$$\mp E_0 \sin\phi \ \cos\phi \ \{ t_p t_p' - t_s t_s' \} \tag{8}$$

For a spherical lens, $\Theta_1' = \Theta_1$ and $\Theta_2' = \Theta_2$ and $t_s t_s' = \cos^2(\Psi/2) t_p t_p'$, and the transmitted intensity pattern $(E_a/E_0)^2$ may be written as the square of

$$\tfrac{1}{4} t_p t_p' \sin 2\phi (1 - \cos\Psi) \tag{9}$$

This result is equivalent to the last equation of Pierscionek and Chan[6].

For a general lens of any shape, which need have only a rotational symmetry axis (coincident with the normal to the polarizer and analyser), we always have

$$t_s = t_p \cos\delta \ \text{and} \ t_s' = t_p' \cos\delta' \tag{10}$$

where $\delta = \Theta_1 - \Theta_2$ and $\delta' = \Theta_1' - \Theta_2'$ are the deviations in direction of the ray at the first and second refractions. Thus

$$t_s t_s' = t_p t_p' \cos\delta \ \cos\delta' \tag{11}$$

and the electric field transmitted through crossed polaroids becomes , from Equation (8),

$$\mp E_0 t_p t_p' \sin\phi \ \cos\phi (1 - \cos\delta \ \cos\delta') \tag{12}$$

The intensity has the $\sin^2\phi \cos^2\phi$ factor (giving a four-leaf clover pattern) for an arbitrary axisymmetric lens. The other factors are independent of the azimuthal angle ϕ, and depend only on the distance of the ray from the axis. For ellipsoids between crossed polaroids, the radial amplitude factor $t_p t_p'(1 \ \cos\delta \cos\delta')$ is shown in *Figure 2*. The Appendix gives details of the calculation of the required angles.

We see that these radial factors have a maximum near the maximum impact parameter, where the rays enter the lens at close to glancing incidence. At grazing incidence the factor drops to zero, since no light enters the lens.

The illustration in Pierscionek's and Charman's articles

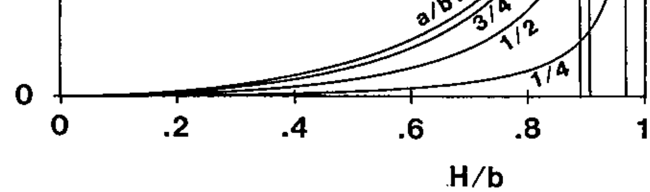

Figure 2. The radial amplitude factor $t_p t_p'(1 - \cos\delta \ \cos\delta')$ for an ellipsoidal lens of index 1.4, as a function of the distance H from the axis of an emergent paraxial ray, drawn for four values of the ratio a/b of the ellipsoidal axes.

are taken with diffuse illumination, and thus correspond to a selection of incident rays, at all angles and all possible entry points on the sphere, which exit approximately parallel to the axis. The analysis given in the Appendix together with Equation (12) can be used to estimate the intensity pattern which is observed in diffuse illumination.

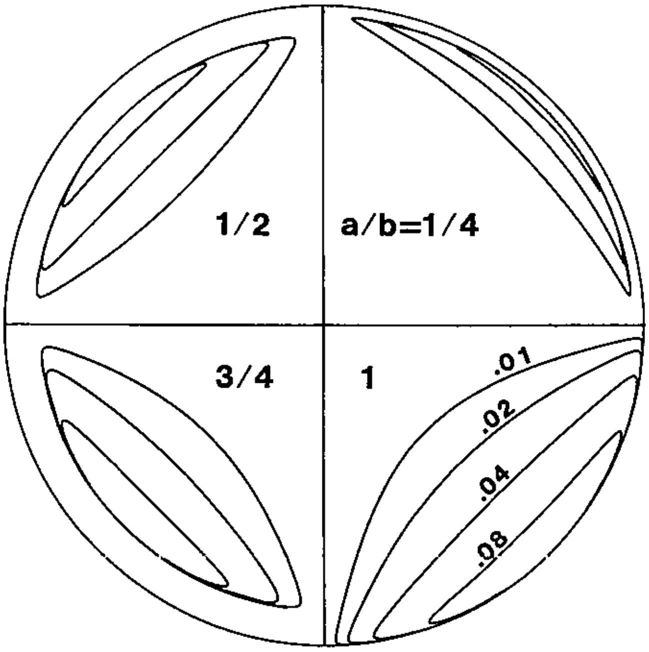

Figure 3. Contours of equal intensity (isogyres) for diffusely illuminated ellipsoidal lenses of refractive index 1.4 between two polaroids. The contours are calculated from Equation (13), and are shown for E_3/E_0 = 0.01, 0.02, 0.04 and 0.08 (the last contour applies to the spherical lens only). Only one quadrant is shown for each of four ellipsoids, with semiaxis ratios a/b = ¼, ½, ¾ and 1. The polarizer and analyser easy axes are vertical and horizontal, or *vice versa*. The radial variable is H/b; note that when $a/b \neq 1$ the maximum value of H is less than the semimajor axis (see the Appendix).

Figure 3 shows contours of equal intensity calculated from the square of the amplitude factor given in Equation (13).

$$E_3/E_0 = \sin\phi \cos\phi \; t_p t_p' (1 - \cos\delta \cos\delta') \qquad (13)$$

We see, on comparison with Figures 3 and 5 of Pierscionek, that the isogyres for a spherical lens closely correspond to experiment, with intensity maxima located near (but not at) the periphery of each quadrant. This is in contrast to the simulation of Pierscionek and Chan, where the intensity maxima occur at about one-third of the distance from the centre to the periphery.

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Appendix

To calculate the amplitude factor $t_p t_p' (1 - \cos\delta \cos\delta')$ we need to know the angles Θ_1, Θ_2, Θ_2' and Θ_1'. Let H be the distance from the axis of the emergent paraxial ray, and let $y(x)$ or $x(y)$ give the shape of the $z = 0$ section through the lens (see *Figure 1*). Let the entry and exit points be A and A′; the relations

$$-\left(\frac{dx}{dy}\right)_{A'} = \tan\Theta_1' \quad , \quad n_1 \sin\Theta_1' = n_2 \sin\Theta_2' \qquad (A1)$$

determine the angles Θ_1' and Θ_2', since A′ = $(x(H), H)$.

Inside the lens the ray is at angle $\Theta_1' - \Theta_2'$ to the horizontal. If the entry point is A = $(x(h), h)$, the slope of the ray is

$$\frac{H - h}{X(H) - x(h)} = \tan(\Theta_1' - \Theta_2') \qquad (A2)$$

This determines h. The tangent to the lens surface at A has slope $(dy/dx)_A$, the normal has slope $-(dx/dy)_A$. Thus

$$\Theta_2 = \Theta_1' - \Theta_2' + \arctan\,(dx/dy)_A \qquad (A3)$$

Finally, Snell's law $n_1\sin\Theta_1 = n_2\sin\Theta_2$ determines Θ_1.

For an ellipsoid of revolution with semiaxes a and b, namely $x^2/a^2+(y^2+z^2)/b^2 = 1$, the $z=0$ section is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \text{ or } x = a\cos\Phi,\ y = b\sin\Phi \quad \text{(A4)}$$

The parametric angle has values $\Phi' = \arcsin(H/b)$ at the exit point A′, and $\Phi = \pi - \arcsin(h/b)$ at the entry point A. The slope of the normal to the lens at the entry point is $-(dx/dy)_A = (a/b)\tan\Phi = -ah/b(b^2-h^2)^{1/2}$. From Equations (A1), (A2) and (A3)

$$\tan\Theta_1' = \frac{a}{b}\tan[\arcsin(H/b)] = \frac{a}{b}\,\frac{H}{(b^2-H^2)^{1/2}} \quad \text{(A5)}$$

$$\Phi = \pi - \Phi' + 2\arctan\left[\frac{a}{b}\right]\tan(\Theta_1' - \Theta_2') \quad \text{(A6)}$$

$$\Theta_2 = \Theta_1' - \Theta_2' - \arctan\left(\frac{a}{b}\tan\Phi\right) \quad \text{(A7)}$$

These equations together with Snell's law determine all the required angles. In the special case of a spherical lens of radius a we regain the results[5,6]

$$\Theta_1' = \arcsin(H/a) = \Theta_1,\ \Theta_2' = \arcsin\left(\frac{n_1H}{n_2a}\right) = \Theta_2 \quad \text{(A8)}$$

Note that $\Phi' = \Theta_1$ and $\Phi = \Theta_1 + \pi - 2\Theta_2$ in the spherical lens case.

All values of H up to the radius are possible in the spherical lens case, but all values of H are not possible in the general case. The limiting value of H for a particular lens is obtained by setting $\Theta_1 = \pi/2$ (the incident ray tangential to the lens surface), and $\Theta_2 = \arcsin(n_1/n_2)$. For larger values of H the reversed ray would be totally reflected within the lens. In practice the maximal value of H is close to b, because for thin lenses ($a \ll b$) the rays are strongly bent only near the periphery of the lens. For an ellipsoidal lens the maximum value of H is a solution of an algebraic equation of the sixth degree, with coefficients depending on a/b and on the ratio of the refractive index of the lens to that of the surrounding medium.