Identities Arising from Two-Cylinder Electrostatics

John Lekner

MacDiarmid Institute and School of Chemical and Physical Sciences
Victoria University of Wellington, PO Box 600,
Wellington, New Zealand
John.Lekner@vuw.ac.nz

Abstract

The electrostatics of two cylinders charged to the same potential has two apparently distinct but equivalent solutions. Their equivalence follows from the uniqueness of the conformal mapping of a multiply connected region onto a circular annulus with concentric slits, up to a rotation of the annulus. Equivalence of the two solutions implies a triply-infinite identity (with two real variables and a real parameter). A special case of the imaginary part of this identity gives a one-parameter series for π.

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1   Two-cylinder electrostatics

Two very different solutions exist to the electrostatic problem of two equal parallel cylinders held at the same potential [6, 2]. The purpose of this note is to show their equivalence, and to make explicit the identities which result from the equivalence.

Figure 1 illustrates the problem and its solution: two parallel conducting cylinders, both of radius a, centered on $(\pm d, 0)$ in the x, y plane. The potential is the same on the two cylinders. The figure shows the equipotentials (solid curves) and field lines (dashed curves).
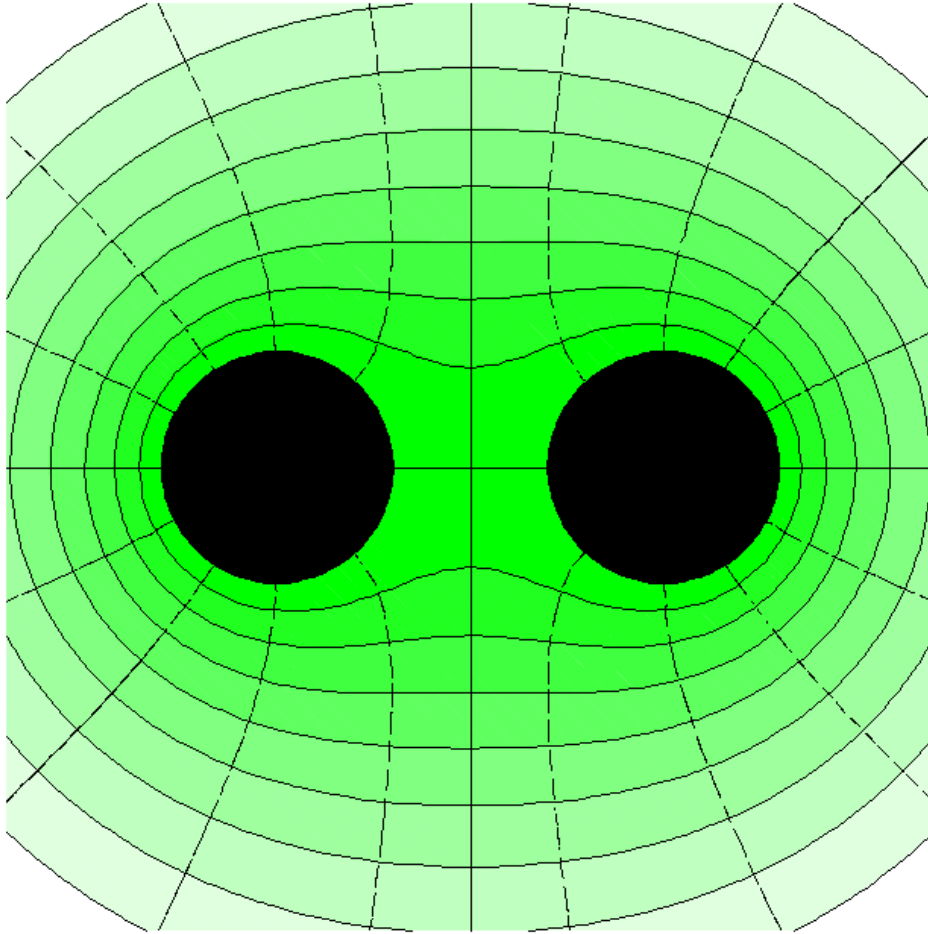
Figure 1: A cross-sectional view of two parallel conducting cylinders charged to the same potential, and the resulting equipotential contours and field lines.

If we normalize the potential on the cylinders to unity, and let λ be the dimensionless ratio in Gaussian units of the total charge per unit length on the two cylinders to the potential on the cylinders, the Darevski [2] potential function becomes $V_D = 1 + \lambda \Phi_D$, where

$$\Phi_D = \ln(2\cosh u - 2\cos v) - u_a + 2\sum_{n=1}^{\infty} \frac{e^{-nu_a}}{n\cosh nu_a}\cosh nu \cos nv \qquad (1)$$

Here u and v are real bicylindrical variables (the two-dimensional analogue of bispherical coordinates; see for example [5], where the same notation is used), related to the Cartesian coordinates x and y of Figure 1 by

$$u = \ln\frac{r_1}{r_2}, \qquad v = \arccos\left(\frac{r_1^2 + r_2^2 - 4\ell^2}{2r_1 r_2}\right), \qquad \ell^2 = d^2 - a^2 \qquad (2)$$

where r_1 and r_2 are the distances from the field point (x, y) to the points

$(-\ell,0)$ and $(\ell,0)$,

$$r_1^2 = (\ell + x)^2 + y^2, \qquad r_2^2 = (\ell - x)^2 + y^2 \tag{3}$$

The positive parameter u_a is determined by the lengths a and d:

$$\sinh u_a = \ell / a \qquad \text{or} \qquad \cosh u_a = d / a \tag{4}$$

The Quilico [6] potential function, in the same bicylindrical coordinates, can be written in the form $V_Q = 1 + \lambda \Phi_Q$, where

$$\Phi_Q = \sum_{n=-\infty}^{\infty} \ln \frac{\tan^2 \dfrac{\pi u}{4u_a} + \tanh^2 \dfrac{\pi}{4u_a}(v + 2n\pi)}{1 + \tan^2 \dfrac{\pi u}{4u_a} \tanh^2 \dfrac{\pi}{4u_a}(v + 2n\pi)} \tag{5}$$

Both Φ_D and Φ_Q satisfy Laplace's equation $(\partial_x^2 + \partial_y^2)\Phi = 0$, and both are zero when $u = \pm u_a$, that is on the two cylinders. That $\Phi_Q(\pm u_a, v) = 0$ is clear from (5), since $\tan^2(\pm \pi / 4) = 1$. In the case of Φ_D, when $u = \pm u_a$ the right-hand side of equation (1) becomes

$$\ln(2\cosh u_a - 2\cos v) - u_a + 2\sum_{n=1}^{\infty} \frac{1}{n} e^{-nu_a} \cos nv \tag{6}$$

To evaluate the sum we shall use the facts that ([4], p97)

$$\cos nv = T_n(c), \qquad \sum_{n=1}^{\infty} T_n(c)t^n = \frac{t(c-t)}{1 - 2ct + t^2}, \qquad |t| < 1 \tag{7}$$

where $c = \cos v$, and the $T_n(c)$ are Chebyshev polynomials of the first kind. Integration of the generating function in (7) gives us

$$\sum_{n=1}^{\infty} T_n(c) \frac{t^n}{n} = -\tfrac{1}{2}\ln(1 - 2ct + t^2), \qquad |t| < 1 \tag{8}$$

In equation (8) we set $t = e^{-u_a}$ to get

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{-nu_a} \cos nv = -\tfrac{1}{2}\ln\left[e^{-u_a}(2\cosh u_a - 2\cos v)\right] \tag{9}$$

which shows that the expression in (6) is zero.

We also need to check that the expressions Φ_D and Φ_Q agree far from the two cylinders. It is then advantageous to use the relations

$$\frac{x + iy}{\ell} = i \cot \frac{v + iu}{2}, \qquad \text{or} \qquad \frac{x}{\ell} = \frac{\sinh u}{\cosh u - \cos v}, \qquad \frac{y}{\ell} = \frac{\sin v}{\cosh u - \cos v} \tag{10}$$

We find from (10) that

$$\frac{r^2}{\ell^2} = \frac{x^2 + y^2}{\ell^2} = \frac{\cosh u + \cos v}{\cosh u - \cos v} = \frac{4}{u^2 + v^2} + \frac{2}{3}\frac{u^2 - v^2}{u^2 + v^2} + O(u^2, v^2) \tag{11}$$

Thus infinity in the x, y plane corresponds to the origin in the u, v plane. For $r \gg \ell$ the leading term in both Φ_D and Φ_Q is logarithmic. In Φ_D we have

$$\ln(2\cosh u - 2\cos v) = \ln(u^2 + v^2) + O(u^2, v^2)$$

$$= 2\ln\frac{2\ell}{r} + O(1) \tag{12}$$

In the Φ_Q infinite sum (5) the dominant term as u and v tend to zero is that with $n = 0$:

$$\ln\frac{\tan^2\dfrac{\pi u}{4u_a} + \tanh^2\dfrac{\pi v}{4u_a}}{1 + \tan^2\dfrac{\pi u}{4u_a}\tanh^2\dfrac{\pi v}{4u_a}} = \ln\left[\left(\frac{\pi}{4u_a}\right)^2(u^2 + v^2) + O(u^4, v^4)\right]$$

$$= 2\ln\frac{2\ell}{r} + O(1) \tag{13}$$

These expressions agree with each other, and with the potential at large distance from a cylinder carrying charge q per unit length, namely $2q\ln(a/r) + O(1)$.

2 Consequent identities

The equality of the potential functions Φ_D and Φ_Q implies the identity

$$\ln(2\cosh u - 2\cos v) - u_a + 2\sum_{n=1}^{\infty}\frac{e^{-nu_a}}{n\cosh nu_a}\cosh nu\cos nv$$

$$= \sum_{n=-\infty}^{\infty}\ln\frac{\tan^2\dfrac{\pi u}{4u_a} + \tanh^2\dfrac{\pi}{4u_a}(v + 2n\pi)}{1 + \tan^2\dfrac{\pi u}{4u_a}\tanh^2\dfrac{\pi}{4u_a}(v + 2n\pi)} \tag{14}$$

There is a related identity, associated with the functions $\Psi(u, v)$ giving the field lines. In each case, the function Ψ is the imaginary part of a complex function $\Phi + i\Psi$. For the Quilico solution, we note that

$$\ln\tan(u + iv) = \tfrac{1}{2}\ln\frac{\tan^2 u + \tanh^2 v}{1 + \tan^2 u\tanh^2 v} + i\arctan\left(\frac{\sinh 2v}{\sin 2u}\right) \tag{15}$$

Thus Φ_Q and Ψ_Q are the real and imaginary parts of a complex analytic function of $u + iv$, namely

$$F_Q(u + iv) = 2\sum_{n=-\infty}^{\infty}\ln\tan\frac{\pi}{4u_a}\left[u + i(v + 2\pi n)\right] \tag{16}$$

For the Darevski solution, we use two functions of $v - iu$,

$$\ln\sin\frac{v - iu}{2} = \tfrac{1}{2}\ln\left[\tfrac{1}{2}(\cosh u - \cos v)\right] + i\arctan\left(\frac{\tanh u/2}{\tan v/2}\right) \tag{17}$$

and

$$\cos n(v - iu) = \cos nv\cosh nu + i\sin nv\sinh nu \tag{18}$$

Thus Φ_D and Ψ_D are the real and imaginary parts of F_D, where

$$F_D(v - iu) = 2\ln 2\sin\frac{v - iu}{2} - u_a + 2\sum_{n=1}^{\infty}\frac{e^{-nu_a}}{n\cosh nu_a}\cos n(v - iu) \tag{19}$$

The equality of $F_D + i\pi/2$ and F_Q gives

$$\ln 2\sin\frac{v - iu}{2} - u_a/2 + \sum_{n=1}^{\infty}\frac{e^{-nu_a}}{n\cosh nu_a}\cos n(v - iu) + i\pi/2$$

$$= \sum_{n=-\infty}^{\infty}\ln\tan\frac{\pi}{4u_a}[u + i(v + 2\pi n)] \tag{20}$$

The real part of (20) gives the identity (14), the imaginary part gives

$$\arctan\left(\frac{\tan v/2}{\tanh u/2}\right) + \sum_{n=1}^{\infty}\frac{e^{-nu_a}}{n\cosh nu_a}\sinh nu\sin nv$$

$$= \sum_{n=-\infty}^{\infty}\arctan\left[\frac{\sinh\dfrac{\pi}{2u_a}(v + 2n\pi)}{\sin\dfrac{\pi nu}{2u_a}}\right] \tag{21}$$

The Quilico imaginary part (the sum on the right-hand side) is not defined when $\sin(\pi nu/2u_a) = 0$. We have to reinterpret the sum over logarithms in (20) as

$$\lim_{N \to \infty}\sum_{-N}^{N}\ln\tan\frac{\pi}{4u_a}(u + iv + 2\pi in) \tag{22}$$

in which the terms with $\pm n$ are taken together, namely

$$\ln\tan\frac{\pi}{4u_a}(u + iv) + \sum_{n=1}^{\infty}\ln\left[\tan\frac{\pi}{4u_a}(u + iv + 2\pi in)\tan\frac{\pi}{4u_a}(u + iv - 2\pi in)\right] \tag{23}$$

Let

$$\tau_n = \tanh\frac{\pi^2 n}{2u_a}, \qquad T_0 = \tan\frac{\pi}{4u_a}(u + iv) = \frac{\tan\dfrac{\pi u}{4u_a} + i\tanh\dfrac{\pi v}{4u_a}}{1 - i\tan\dfrac{\pi u}{4u_a}\tanh\dfrac{\pi v}{4u_a}} \tag{24}$$

Expanding the tangents in (23) gives for the right-hand side of (20)

$$\ln T_0 + \sum_{n=1}^{\infty}\ln\frac{T_0^2 + \tau_n^2}{1 + T_0^2\tau_n^2} \tag{25}$$

For large $n, \tau_n \to 1 - 2\exp(-\pi^2 n/u_a)$, and hence the series in (25) converges exponentially with n.

3 Special cases

The central identity (20) is valid for real u and v, $|u| < u_a$, and $u_a > 0$. We

discuss some interesting special cases. By construction, the real parts of both sides in (20) are zero on $u = \pm u_a$. The imaginary part of (20) on $u = u_a$ is

$$\arctan\left(\frac{\tan v/2}{\tanh u_a/2}\right) + \sum_{n=1}^{\infty} \frac{1}{n} e^{-nu_a} \tanh nu_a \sin nv$$

$$= 2\arctan\left(\tanh\frac{\pi v}{4u_a}\right) + 2\sum_{n=1}^{\infty} \arctan\left(\frac{\sinh\dfrac{\pi v}{2u_a}}{\cosh\dfrac{\pi^2 n}{u_a}}\right) \tag{26}$$

The first term on the left of (26) needs to be replaced for $\pi < v < 2\pi$ by the full form $\pi/2 - \arctan\left(-\sinh(u_a/2)\cos(v/2), \cosh(u_a/2)\sin(v/2)\right)$, where $\arctan(A, B)$ is the arctangent of A/B, placed in the correct quadrant according to the signs of A and B. The two sides of (26) are manifestly zero when $v = 0$. At $v = \pi$ we get the identity

$$\frac{\pi}{4} = \arctan\left(\tanh\frac{\pi^2}{4u_a}\right) + \sum_{n=1}^{\infty} \arctan\left(\frac{\sinh\dfrac{\pi^2}{2u_a}}{\cosh\dfrac{\pi^2 n}{u_a}}\right) \tag{27}$$

If we set $\pi^2/2u_a = \ln\alpha$ $(\alpha > 1)$, the equality (27) reads

$$\frac{\pi}{4} = \arctan\frac{\alpha - 1}{\alpha + 1} + \sum_{n=1}^{\infty} \arctan\left(\frac{\alpha - \alpha^{-1}}{\alpha^{2n} + \alpha^{-2n}}\right) \tag{28}$$

Specializing further to $u_a = \pi^2/4$ $(\alpha = e^2)$ gives

$$\frac{\pi}{4} = \arctan(\tanh 1) + \sum_{n=1}^{\infty} \arctan\left(\frac{\sinh 2}{\cosh 4n}\right) \tag{29}$$

There is an infinity of such special cases, of course. All of the identities, general and special, appear to be new.

4   Discussion

The identities we have presented follow from the uniqueness of the mapping of multiply connected regions on a circular annulus with concentric slits (Ahlfors [1] Section 5.1, p248; Henrici [3] Volume 3, Section 17.6). The solution of the Dirichlet problem is unique up to a rotation of the annulus (equivalently: up to a factor of unit modulus). It would be interesting to have a direct proof of the general identity, or even of the sub-identities obtained by taking special values of the variables u and v and parameter u_a.

References

[1]   L. V. Ahlfors, Complex analysis (2ed), McGraw-Hill, New York, 1966.

[2]   A. I. Darevski, The electrostatic field of a split phase, [in Russian] Elektrichestvo, 9 (1958), 16-19.

[3]   P. Henrici, Applied and computational complex analysis, Wiley, New York, 1993.

[4]   N. N. Lebedev, Special functions and their applications, Dover, New York, 1972.

[5]   J. Lekner, Near approach of two conducting spheres: enhancement of external electric field, J. Electrostatics, 69 (2011), 559-563.

[6]   G. Quilico, Campo elettrico di un conduttore bifilare, L'Elettrotecnica, 41 (1954), 530-538.