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Electrostatic calibration of sphere–sphere forces

John Lekner

The MacDiarmid Institute for Advanced Materials and Nanotechnology, School of Chemical and Physical Sciences, Victoria University of Wellington, PO Box 600, Wellington, New Zealand

E-mail: john.lekner@vuw.ac.nz

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Abstract

We give formulae for the calibration of a force apparatus based on sphere–sphere geometry, both exact and approximate. The expressions are for the electrostatic force between conducting spheres of radii a and b, held at potentials V_a and V_b, as a function of the separation s between them. The force, which can be attractive or repulsive, is given to parts-per-thousand accuracy for $s \lesssim a$ by a simple formula when the sphere radii are equal. When $V_a \neq V_b$, the dominant force term at close approach is attractive, with magnitude $(V_a - V_b)^2 \frac{ab}{4(a+b)s}$. When $V_a = V_b$ the force is repulsive, taking the Kelvin contact value $V_a^2(4\ln 2 - 1)/24$ when $a = b$.

Keywords: sphere, electrostatic force, calibration

(Some figures may appear in colour only in the online journal)

1. Introduction

The purpose of this paper is to provide simple formulae for the calculation of electrostatic force between two conducting spheres held at constant potentials V_a and V_b. The results can be used in the calibration of force apparatus. In more general geometries the calculation of electrostatic force is difficult (see for example [1]), but for the sphere–sphere geometry Wistrom and Khachatourian [2] and Saranin and Mayer [3] have provided theoretical and experimental results. Wistrom and Khachatourian express the force between the spheres in terms of the surface charge densities on the spheres, which in turn satisfy coupled integral equations, to be solved numerically. Saranin and Mayer consider the case of conducting spheres kept at the *same* potential (their section 2). The force between spheres kept at a constant potential difference has been treated earlier [4]. Here we give an exact expression for the force between spheres kept at fixed arbitrary potentials, based on known results for the capacitance coefficients. There follows a simple but accurate analytical approximation for the force between two conducting spheres, which becomes exact in the limit of close approach.

2. Electrostatic force between two spheres each kept at a fixed potential

Let the spheres have radii a and b, and let the distance between their centres be c. The nearest points of the spheres are separated by $s = c - a - b$; potentials on the two spheres are maintained at V_a and V_b by constant-voltage sources, designated as batteries in figure 1.

The electrostatic energy of the system depicted in figure 1 is

$$W = \tfrac{1}{2}Q_aV_a + \tfrac{1}{2}Q_bV_b - Q_aV_a - Q_bV_b$$
$$= -\tfrac{1}{2}Q_aV_a - \tfrac{1}{2}Q_bV_b. \qquad (1)$$

(The reason for the subtraction of Q_aV_a and Q_bV_b is that the energy of a battery is decreased by QV on supplying charge Q at potential V; see for example [5].) The charges Q_a and Q_b on the two spheres are determined by the potentials V_a and V_b and the capacitance coefficients [6] C_{aa}, C_{ab} and C_{bb}:

$$Q_a = C_{aa}V_a + C_{ab}V_b, \qquad Q_b = C_{ab}V_a + C_{bb}V_b. \qquad (2)$$

Thus the electrostatic energy of the system is given by

$$W = -\tfrac{1}{2}(C_{aa}V_a^2 + 2C_{ab}V_aV_b + C_{bb}V_b^2). \qquad (3)$$

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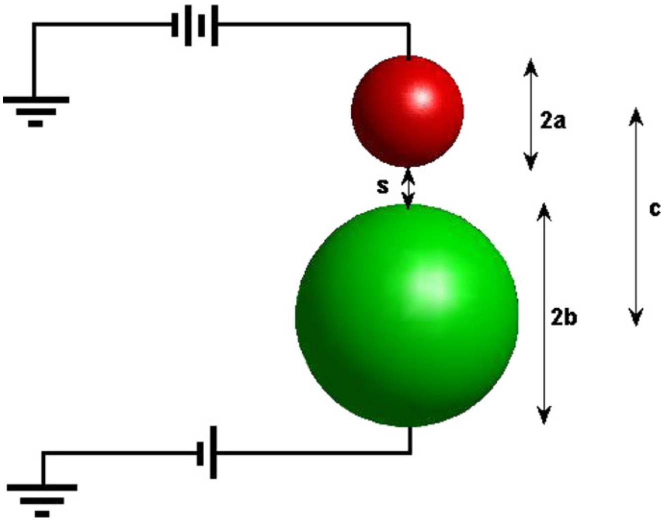

Figure 1. Two conducting spheres of radii a and b, held at constant potentials V_a and V_b (relative to earth). The separation between the spheres is s, the centre-to-centre distance is $c = a + b + s$.

When the conductors are spheres, the capacitance coefficients are given exactly by the infinite sums [6–9]

$$C_{aa} = ab \sum_{n=0}^{\infty} \frac{\sinh U}{a \sinh nU + b \sinh(n+1)U}$$
$$C_{bb} = ab \sum_{n=0}^{\infty} \frac{\sinh U}{b \sinh nU + a \sinh(n+1)U} \qquad (4)$$
$$C_{ab} = -\frac{ab}{c} \sum_{n=1}^{\infty} \frac{\sinh U}{\sinh nU}.$$

The dimensionless parameter U is defined by

$$\cosh U = \frac{c^2 - a^2 - b^2}{2ab} = 1 + s\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{s^2}{2ab}. \qquad (5)$$

3. Electrostatic force between two spheres at potentials V_a and V_b

The above results show that when the potentials V_a, V_b and the lengths a, b and c (or s) are specified, the energy W is determined. The electrostatic force between the spheres is given by

$$F = -\partial_c W \quad \text{or} \quad F = -\partial_s W. \qquad (6)$$

The sums in (4) converge rapidly except in the limit of close approach, when s is small compared to $a + b$. Then U is small compared to unity, and convergence of the sums (4) becomes progressively slower as $s/(a + b)$ tends to zero. Fortunately, one can derive accurate analytical approximations in the close-approach limit [7, 10–12]. To lowest order in s, these are

$$C_{aa} = \frac{ab}{a+b}\left\{\frac{1}{2}\ln\left[\frac{2ab}{(a+b)s}\right] - \psi\left(\frac{b}{a+b}\right) + O(s)\right\}$$
$$C_{bb} = \frac{ab}{a+b}\left\{\frac{1}{2}\ln\left[\frac{2ab}{(a+b)s}\right] - \psi\left(\frac{a}{a+b}\right) + O(s)\right\} \qquad (7)$$
$$C_{ab} = -\frac{ab}{a+b}\left\{\frac{1}{2}\ln\left[\frac{2ab}{(a+b)s}\right] + \gamma + O(s)\right\},$$

where $\psi(z) = \mathrm{d}\ln\Gamma(z)/\mathrm{d}z$ is the logarithmic derivative of the gamma function, and $\gamma = -\psi(1) = 0.5772...$ is Euler's constant. Differentiation of the logarithmic terms gives rise to a force inversely proportional to the separation at close approach:

$$F = -\frac{ab}{4(a+b)}(V_a - V_b)^2\frac{1}{s} + O(1). \qquad (8)$$

We shall give the $O(1)$ term, and the $O(s)$ term for the $a = b$ case later. Here we just note that Russell's expression for the $a = b$ force [6] is in agreement with (8):

$$F_R = -\frac{1}{8}(V_a - V_b)^2\frac{a}{s} + \frac{1}{48}(V_a - V_b)^2\left[\ln\frac{a}{s} + 2\gamma + \frac{1}{6}\right]$$
$$+ \frac{1}{48}\left(V_a^2 + V_b^2\right)(4\ \ln 2 - 1) + O(s). \qquad (9)$$

(Russell gives part of the $O(s)$ term also.) The graphs of force in figures 3(*b*) and (*c*) of Wistrom and Khachatourian [2] are consistent with $F \sim s^{-1}$ at short range, and the force between conducting spheres with fixed charges Q_a and Q_b is also of this form, *unless* the ratio of the charges is the same as would be obtained by bringing the spheres into contact, in which case the force is repulsive and non-singular [12]. The third term in (9) is a constant force of repulsion, dominant for $V_a = V_b$ in the $a = b$ case, and was first obtained by Kelvin in 1853 [13]. (The 1853 Kelvin formula has an interesting history: see section 4 of [12], and [15]).

We return to the general ($a \neq b$) case: from the form of (3) and from (8) it follows that the force $F = -\partial_s W$ between the spheres is of the form

$$F = \tfrac{1}{2}D_{aa}V_a^2 + D_{ab}V_aV_b + \tfrac{1}{2}D_{bb}V_b^2, \qquad (10)$$

where the coefficients D_{ij} are derivatives of the capacitance coefficients:

$$D_{aa} = \partial_s C_{aa}, \quad D_{ab} = \partial_s C_{ab}, \quad D_{bb} = \partial_s C_{bb}. \qquad (11)$$

(The expressions for force are in Gaussian units, in which the dimensions of force are the same as that of (potential)^2, and the dimension of capacitance is length. In SI units the force is obtained by multiplying the expressions given here by $4\pi\varepsilon_0$.) On using expressions for the capacitance coefficients valid to $O(s)$ [7, 10, 12] we obtain the force coefficients to $O(1, \ln s)$

$$D_{aa} = -\frac{ab}{2(a+b)s} + \frac{A + B\psi'(\beta) + C\,[2\psi(\beta) - L]}{36(a+b)^3}$$
$$D_{bb} = -\frac{ab}{2(a+b)s} + \frac{A - B\psi'(1-\beta) + C\,[2\psi(1-\beta) - L]}{36(a+b)^3}$$
$$D_{ab} = \frac{ab}{2(a+b)s}$$
$$+ \frac{(2a-b)(2b-a) + 6(a^2+ab+b^2)[2\gamma + L]}{36(a+b)^3}, \qquad (12)$$

where we have used the shorthand notations

$$A = (a+b)(2a^2 - ab + 2b^2), \quad B = 12ab(a-b)$$
$$C = 6(a^3 + b^3), \quad L = \ln\left[\frac{2ab}{(a+b)s}\right], \quad \beta = \frac{b}{a+b}. \qquad (13)$$

The equations (10) to (13) give an analytical approximation to the force between spheres of unequal size, to the next order up from (8). When $a = b$ these expressions inserted into (10) give Russell's result (9).

To obtain terms of $O(s)$ in the force, we need terms of order U^2 in D_{aa}, D_{ab} and D_{bb}. These may be obtained by taking derivatives of the (exact) integral forms of the capacitance coefficients given in [11]. In the general case the results are somewhat cumbrous [12], but when $a = b$ the electrostatic force between two spheres becomes

$$F = F_R - \frac{(V_a - V_b)^2}{720}\left[\ln\frac{a}{s} + 2\gamma - \frac{169}{40}\right]\frac{s}{a} - \frac{V_a^2 + V_b^2}{720}\left[4\ln 2 + \frac{17}{4}\right]\frac{s}{a} + O(s^2). \quad (14)$$

The derivation proceeds from the force expression (10) which defines the force in terms of the derivatives of the capacitance coefficients, (11). The capacitance coefficients of equation (4.7) of [12] are in terms of the variable U defined in (5). We convert from derivatives with respect to s (or c) to derivatives with respect to U as in equation (4.9) of [12]. The general result contains $\psi(z) = \mathrm{d}\ln\Gamma(z)/\mathrm{d}z$ and $\psi'(z) = \mathrm{d}^2\ln\Gamma(z)/\mathrm{d}z^2$, but when $a = b$ the use of

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\,\ln 2, \qquad \psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2} \quad (15)$$

([14], equations (6.3.3) and (6.4.4)) simplifies the derivatives to

$$D_{aa} = -\frac{a}{4s} + \frac{1}{24}\left[\ln\frac{a}{s} + 2\gamma + 4\ln(2) - \frac{5}{6}\right] - \frac{s}{a}\frac{1}{360}\left[\ln\frac{a}{s} + 2\gamma + 4\ln(2) + \frac{1}{40}\right] \quad (16)$$

(to order s/a); $D_{bb} = D_{aa}$, and

$$D_{ab} = \frac{a}{4s} - \frac{1}{24}\left[\ln\frac{a}{s} + 2\gamma + \frac{1}{6}\right] + \frac{s}{a}\frac{1}{360}\left[\ln\frac{a}{s} + 2\gamma - \frac{169}{40}\right]. \quad (17)$$

Insertion of these values of the capacitance derivatives into the force expression (10) gives the result (14).

4. Illustrative results and discussion

Figure 2 shows the force curves for the same parameters as used by Wistrom and Khachatourian [2] in their figures 3(*a*), (*b*) and (*c*). When $V_b = V_a$ the force is repulsive at all distances, otherwise the force becomes attractive (negative) when the approach is close enough. (For $V_b/V_a = 51/55$ and $40/55$ this occurs at $s/a = 0.009\,559$ and $0.157\,465$, respectively.) Attraction occurs between spheres both at a positive potential because of mutual polarization: for spheres with equal radii, the sphere at the lower potential obtains a negatively charged region neighbouring the other sphere (see figure 2 of [2], and figure 7 of [12]). Attraction also happens between spheres that have fixed like charges, except when those charges are in the ratio that would obtain by the spheres being brought into contact [12].

The exact force (full curves) is found by differentiating the exact expressions for the capacitance coefficients given

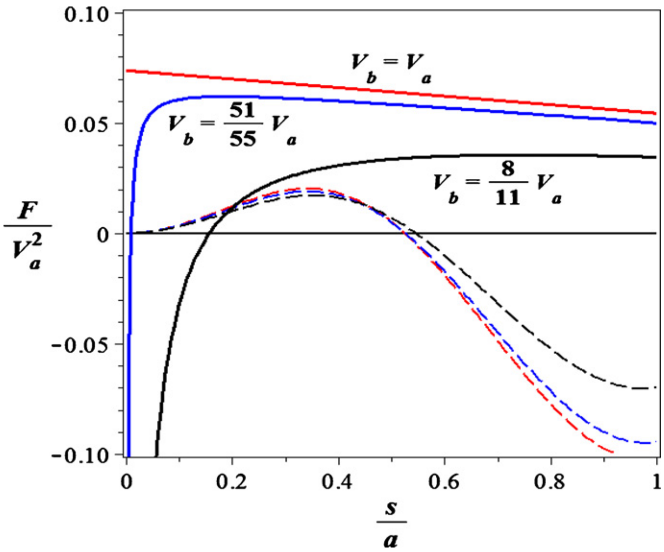

Figure 2. The electrostatic force between spheres of radius a at potentials V_a and V_b, versus s/a (s is the separation of the closest points of the spheres). The three plots show F/V_a^2, for the same potential ratios as used in figure 3 of [2], namely $V_b/V_a = 1, 51/55$ and $40/55$. (In SI units the plots are of $F/(4\pi\varepsilon_0 V_a^2)$ versus s/a.) The dashed curves give the difference between the exact and approximate values of F/V_a^2, times 500. For example, the value -0.05 corresponds to (approximate–exact) $F/V_a^2 = 0.0001$, which is $1/100$ of the smallest scale division shown.

in (4), with $D_{ij} = \partial_c C_{ij}$ and $\sinh U \partial_c U = c/ab$ from (5). The approximate curves calculated from (14) are barely distinguishable from the exact curves over the physically important range $0 \leqslant s \leqslant a$: even at $s = a$ the force values differ by only four parts per thousand. The dashed curves give 500 times the difference between the exact and approximate values of F/V_a^2.

In summary: we have given simple analytic formulae which allow easy calculation of the electrostatic force between conducting spheres held at potentials V_a and V_b. We hope these formulae will facilitate the electrostatic calibration of sphere–sphere force apparatus.

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