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John Creedy and Norman Gemmell

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Further enquiries to:
The Administrator
Chair in Public Finance
Victoria University of Wellington
PO Box 600
Wellington 6041
New Zealand

Phone: +64-4-463-9656
Email: cpf-info@vuw.ac.nz

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Types of Income Mobility: Insights from TIM Curves*

John Creedy and Norman Gemmell†

Abstract

Jenkins and Lambert (1997) demonstrated that a number of measures of poverty could be combined and compared using the "Three Is of Poverty" (TIP) curve; the 'three Is' being the incidence, intensity and inequality of poverty. This paper takes the insights from the TIP curve and applies them to measures of income mobility, proposing a "Three Is of Mobility", or TIM, curve. The analysis is then applied to the concept of poverty persistence. Illustrations are provided using income data from random samples of New Zealand income taxpayers over the period 1998 to 2010.

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†Victoria Business School, Victoria University of Wellington.

1 Introduction

When comparing distributions of non-negative cross-sectional economic variables, such as annual income or consumption, the Lorenz curve is ubiquitous. With individual observations arranged in ascending order, this plots (within a box of unit height and base) the cumulative proportion of total income (the first-moment distribution function) against the corresponding cumulative proportion of individuals or units (the distribution function). A normalised area measure of the distance between the Lorenz curve and the line of equality gives rise to the equally famous Gini inequality measure.¹ Furthermore, the concept of ‘Lorenz dominance’ provides an immediate qualitative comparison between the inequality of two distributions, and this can be given a welfare interpretation when combined with the value judgement summarised by the ‘principle of transfers’. The Lorenz curve thus provides a valuable diagrammatic summary, providing much more information than either the density function or the distribution function alone.²

Where concern is largely for those towards the lower end of the distribution – those below a poverty line – an alternative diagrammatic device involves, for incomes again arranged in ascending order, plotting the cumulative (absolute) poverty gap per person against the corresponding cumulative proportion of people. This gives rise to a TIP curve, named by Jenkins and Lambert (1997) for its ability to indicate the ‘Three "I"s of Poverty’, namely incidence, intensity and inequality. As with the Lorenz curve, dominance properties hold and the curve is a straight line (for those below the poverty line) in situations where all the poor have equal incomes (or consumption, whichever metric is applicable). In the case of Lorenz and TIP curves, comparisons involving intersecting curves lead to the need to impose more structure on evaluations, in the form of particular value judgements

¹This measure can be related to an explicit social welfare function involving a ‘reverse-rank’ weighted average of incomes and an inequality measure based on an ‘equally distributed equivalent income measure.’

²The so-called Pen Parade is simply the distribution function rotated through 90 degrees, therefore showing income on the vertical axis and the cumulative proportion of people on the horizontal axis. It is used, along with the metaphorical parade of individuals aligned from poor to rich, mainly in popular presentations. See Pen (1971).

and quantitative inequality and poverty measures.

A challenge arises in the context of income mobility, where there is a need for a type of diagram which can capture, in an easily perceived way, the incidence, intensity and inequality of mobility and, furthermore, can give rise to dominance results. There has been much discussion regarding the types of, and meaning attached to, mobility.³ A variety of measures exist in the literature, often measuring different concepts.

Motivated by a desire to illustrate and summarise the information contained in a transition matrix, Trede (1998) concentrated on the conditional distributions of income (relative to, say, the median) in one year, given incomes in an earlier year. He proposed the use of diagrams showing profiles of various quantiles of the conditional distributions, with relative income in the initial year on the horizontal axis.⁴

An alternative approach, using cross-sectional data for two periods, is to produce a ‘growth incidence curve’ (GIC). This plots the growth rate between two periods of each quantile or percentile of the distribution in the initial period. This can easily display relative growth differences, by subtracting the overall income growth.⁵

The present paper shows that a more informative diagram is obtained, using longitudinal data, by instead plotting the cumulative proportional income change per capita, with individuals ranked in ascending order, against the corresponding proportion of people. Such a diagram clearly bears a close resemblance to the TIP curve and, importantly, separately identifies the incidence, intensity and inequality dimensions of mobility. It is therefore called a ‘Three "I"s of Mobility’, or TIM curve. In addition, the TIM curve concept can also be applied to positional change (re-ranking) measures of mobility.

³Jäntti and Jenkins (2015) categorise the wide range of mobility concepts into four types. Mobility as (i) ‘positional change’; (ii) ‘individual income growth’; (iii) ‘with reference to its impact on inequality in longer-term incomes’; and (iv) ‘income risk’. As they note, Jäntti and Jenkins’ fourth concept of income risk can be regarded as an aspect of concept (iii) - where changes in an income inequality measure over time have both a permanent predictable, and a transitory unpredictable component. The latter is associated with income risk.

⁴Further details are given in Appendix A.

⁵Bourguignon (2011) draws a distinction between the GIC and a corresponding curve based on longitudinal data, which he calls a ‘non-anonymous growth incidence curve’.

In Section 2, a brief summary of the TIP curve is given. The TIM curve is then defined and its properties examined in Section 3. Section 4 examines poverty persistence. Empirical evidence for New Zealand individuals is presented in Section 5. Instead of considering the extent of income changes, Section 6 considers positional mobility, which is concerned with the rank-order changes of individuals in the distribution. Reranking measures for New Zealand are reported in Section 7. Conclusions are in Section 8.

2 The TIP Curve

Let x_i denote individual i 's income, with $i = 1, \dots, n$. Where x_p is the poverty line, the poverty gaps are defined by $g(x_i) = 0$ for $x_i > x_p$ and $g(x_i) = x_p - x_i$ for $x_i < x_p$. When incomes are ranked in ascending order, the TIP curve is obtained by plotting $\frac{1}{n} \sum_{i=1}^k g(x_i)$ against $\frac{k}{n}$ for $k = 1, \dots, n$. That is, the total cumulative poverty gap per capita is plotted against the associated proportion of people.

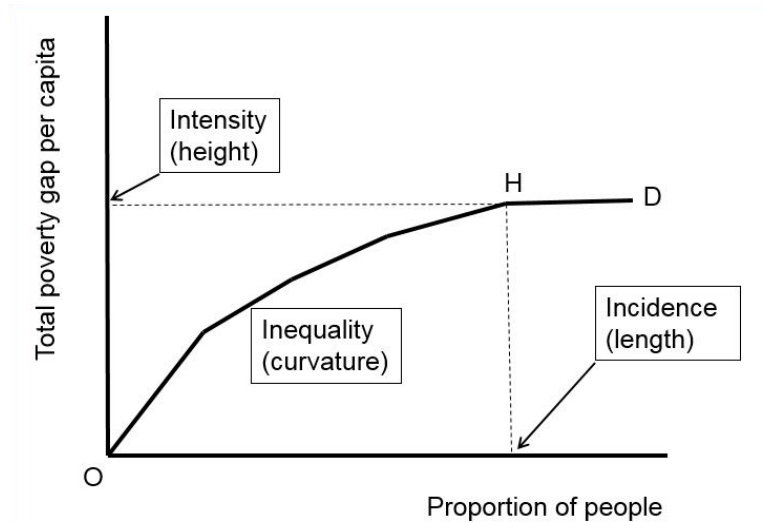


Figure 1: A TIP Curve

An example is shown in Figure 1. The slope at any point is equal to the average poverty gap; flattening shows the extent to which the average

poverty gap falls as income rises towards y_p ; hence inequality among the poor is reflected in the curvature of the TIP curve. The curve is horizontal beyond H , since this fraction of the population is not in poverty. Poverty can be said to be unambiguously higher where a TIP curve lies wholly above and to the left of an alternative TIP curve.

3 The TIM Curve

A ‘Three "I"s of Mobility’ (TIM) curve can be defined as follows. Define $y_i = \log x_i$ as the logarithm of income of person i , for $i = 1, \dots, n$. Hence $y_{i,t} - y_{i,t-1}$ is (approximately) person i ’s proportional change in income from period $t - 1$ to t . With incomes ranked in ascending order, plot $\frac{1}{n} \sum_{i=1}^k (y_{i,t} - y_{i,t-1})$ against $\frac{k}{n}$, for $k = 1, \dots, n$. Thus the TIM curve plots the cumulative proportional income change per capita against the corresponding proportion of individuals. An example is shown in Figure 2. This curve reflects a situation in which relatively lower-income individuals receive proportional income increases which are greater than that of average (geometric mean) income.

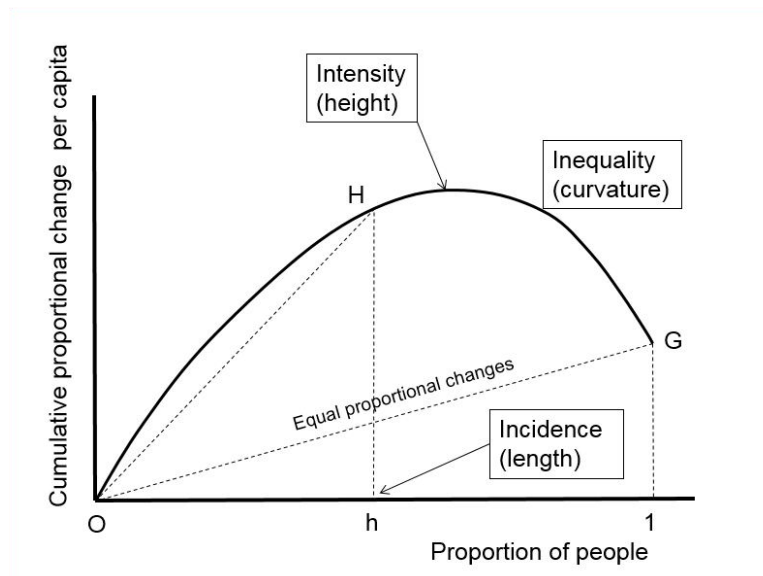


Figure 2: A TIM Curve

If all incomes were to increase by the same proportion, the TIM curve would be the straight line OG. The height, G, indicates the average growth rate of the population as a whole. Furthermore, just as with the TIP curve, ‘inequality’ is reflected in the degree of curvature. However, there is a potential ambiguity here, since the TIP curve refers to a cross-sectional distribution whereas the TIM curve refers to income changes. To avoid confusion over nomenclature, when referring to the ‘inequality dimension’ of mobility (one of the three “T”s), the term ‘interpersonal dispersion’ of mobility will instead be used. The term ‘inequality’ is henceforth used only in reference to the inequality of incomes in a cross-sectional distribution, unless otherwise stated.

Suppose interest is focussed on a proportion of the population, for example those below the h th percentile, as indicated in Figure 2. There is less interpersonal dispersion of income changes among the group below h , shown by the fact that the TIM curve from O to H is closer to a straight line than the complete curve OHG. The TIM curve shows that the income growth of those below h is larger than that of the population as a whole. The average growth rate among the poor (the intensity of their growth) is given by the height, H.

If concern is largely for those below a poverty line, x_p , the corresponding percentile is $h_p = H(x_p)$, where $H(x)$ is the distribution function of x . The TIM curve gives an immediate indication of whether income changes have been ‘pro-poor’.⁶

Figure 3 illustrates a TIM curve reflecting a very different pattern of mobility. In this case the lower-income groups experience smaller proportional increases in income than those with higher incomes. If h_p is to the right of the intersection of the TIM curve with the line OG, average growth of those in poverty exceeds overall growth. Yet this reflects quite different experiences among the poor.

The TIM curve can be examined more formally as follows. As above, $y = \log(x)$ denotes the logarithm of income. For convenience, suppose incomes are described by a continuous distribution where $H(x_t)$ and $F(y_t)$ denote

⁶Pro-poor growth using the GIC curve is examined in Appendix B.

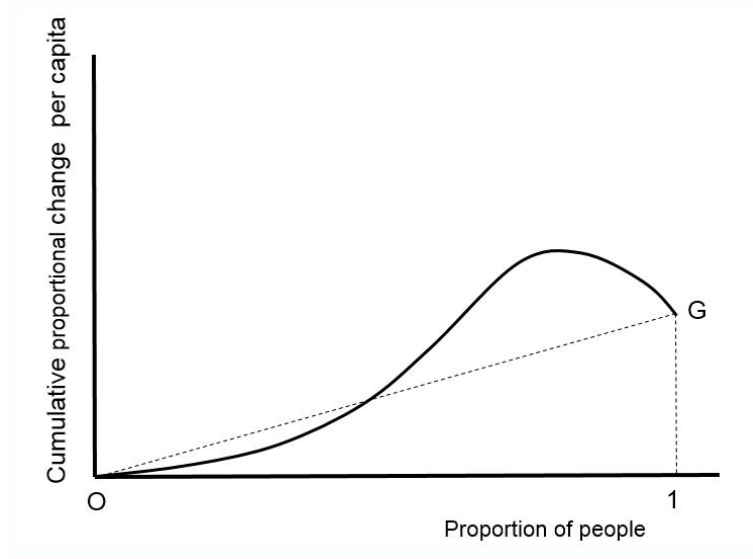


Figure 3: Example of Differential Mobility Pattern

respectively the distribution functions of income and log-income at time t , with population size, n . For incomes ranked in ascending order, the TIM curve plots the cumulative proportional income changes, $y_t - y_{t-1}$, per capita, denoted $M_{h,t}$, against the corresponding proportion of people, h , where:

$$h = F(y_{h,t-1}) \quad (1)$$

Thus $y_{h,t-1} = F^{-1}(h)$ is the log-income corresponding to the h^{th} percentile. Hence, the TIM curve plots $M_{h,t}$, given by:

$$M_{h,t} = \int_0^{y_{h,t-1}} (y_t - y_{t-1}) dF(y_{t-1}) \quad (2)$$

against h .⁷

Let μ_t denote the arithmetic mean of log-income (that is, the logarithm of the geometric mean, G_t , of income, x_t). Then equation (2) can be written

⁷For very large datasets it is convenient to plot values of the cumulative proportional change corresponding to percentiles, P_j , for $P_1 = 0.01$ and $P_j = P_{j-1} + 0.01$, for $j = 2, \dots, 100$. Thus, obtain the cumulative sum $M_j = \frac{1}{n} \sum_{i=1}^{nP_j} (y_{i,t} - y_{i,t-1})$, where as above n is the number of individuals in the sample. Hence for $j = 2, \dots, 100$: $M_j = M_{j-1}/n + \frac{1}{n} \sum_{i=nP_{j-1}+1}^{nP_j} (y_{i,t} - y_{i,t-1})$. The TIM curve is then plotted using just 100 values.

as:

$$M_{h,t} = \int_0^{y_{h,t-1}} \{(y_t - \mu_t) - (y_{t-1} - \mu_{t-1})\} dF(y_{t-1}) + (\mu_t - \mu_{t-1}) F(y_{h,t-1}) \quad (3)$$

The term, $y_t - \mu_t$ is equal to $\log(x_t/G_t)$. Hence $(y_t - \mu_t) - (y_{t-1} - \mu_{t-1})$ is the proportional change in *relative* income. Thus, $M_{h,t}$ consists of the cumulative proportional change in income *relative* to the geometric mean, *plus* a component that depends only on the proportional change in geometric mean income.

If all individuals receive exactly the same relative income change, g , say, then relative positions are unchanged and $(y_t - \mu_t) - (y_{t-1} - \mu_{t-1}) = 0$, with $(\mu_t - \mu_{t-1}) F(y_{h,t-1}) = gh$, so that $M_{h,t}$ plotted against h is simply a straight line with a slope of g . This means that the nature of mobility – the extent to which it is equalising or disequalising over any range of the income distribution – can be seen immediately by the extent to which the TIM curve deviates from a straight line. Appendix E considers a special case where there is a systematic equalising tendency (regression to the geometric mean), along with a stochastic component; this case has received substantial attention in studies of income dynamics.

4 Poverty Persistence

A natural question to ask is whether the TIM curve can illustrate poverty persistence, reflecting the extent to which upward income mobility shifts individuals from below, to above, a given income poverty threshold, x_p ? In fact, because the TIM curve illustrates the *cumulative* extent of mobility for those below h_p , it is not particularly helpful in illustrating the ability of *individuals* below h_p to escape from poverty. Nevertheless, if poverty is measured in relative terms and the TIM curve up to h_p lies on or below the straight line OG, as in Figure 3 for example, then income growth for those below h_p as a whole, is insufficient to lift this group in poverty at $t - 1$ above the poverty line at t . That is, had the income growth experienced in aggregate by those below h_p been redistributed among those individuals to maximise

the numbers above x_p at t , there is no reallocation that could have lifted all of them out of poverty.

Nevertheless, many individuals within this group at $t - 1$ may experience sufficient income growth between $t - 1$ and t to raise their income levels above x_p . It is assumed here that the poverty income threshold is constant in both years. The relevant condition, shown in Appendix D is, for those individuals for whom $x_{i,t-1} < x_p$:

$$g_i > \frac{x_p}{x_{i,t-1}} - 1 \quad (4)$$

where $g_i = \frac{dx_i}{x_{i,t-1}}$ is individual i 's proportional income growth between $t - 1$ and t . More generally, all individuals can be allocated to one of four groups based on their values of g_i and $x_{i,t-1}$. These are given in Table 1:

	in poverty	out of poverty
move	$g_i < \frac{x_p}{x_{i,t-1}} - 1;$ $x_{i,t-1} > x_p$	$g_i > \frac{x_p}{x_{i,t-1}} - 1;$ $x_{i,t-1} < x_p$
persist	$g_i < \frac{x_p}{x_{i,t-1}} - 1;$ $x_{i,t-1} < x_p$	$g_i > \frac{x_p}{x_{i,t-1}} - 1;$ $x_{i,t-1} > x_p$

The two conditions separating the four groups therefore are $g^* = \frac{x_p}{x_{i,t-1}} - 1$ and $x_{i,t-1}^* = x_p$. These can be illustrated by a variant of the TIM curve. Whereas the previously discussed TIM curve plots (for incomes in ascending order) cumulative proportional income changes per capita against the corresponding proportion of people, h , in this case individual income growth rates, g_i and g^* , may be plotted against h , for any given income poverty threshold, x_p , and associated h_p .

Figure 4 shows the values of g^* against h , for a poverty income threshold, x_p , such that $h_p = 0.2$ (20%), and the g^* profile crosses the x -axis at $h = 0.2$. To the left of $h = 0.2$ actual growth rates greater than g^* are sufficient to move the individual out of poverty; that is, above h_p . Conversely, to the right of $h = 0.2$ actual growth rates less than g^* are sufficiently negative to move the individual into poverty, below h_p .

It can be seen that the poverty persistence profile, g^* , asymptotes at -1.0

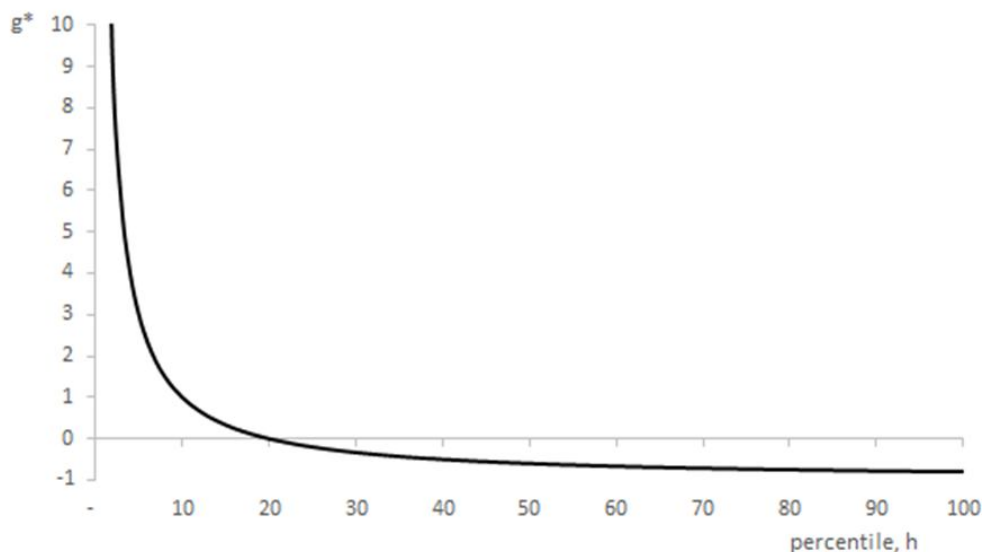


Figure 4: Poverty Persistence

(-100%); that is, as incomes become very large relative to x_p , the required (negative) growth rate to move such individuals into poverty approaches -100%. In the next section, these TIM and poverty persistence curve concepts are applied to panel unit record income data for New Zealand.

5 TIM and Poverty Persistence Curves for New Zealand

5.1 The data

The data used to illustrate the TIM and poverty persistence curves are based on a 2% random sample of individual New Zealand Inland Revenue personal income taxpayers. Using data for 1998, 2002, 2006 and 2010, three separate panels were obtained for 1998-2002, 2002-06 and 2006-10. Each (5-year) panel contains incomes for both years for the same taxpayers.

To avoid the exercise being contaminated by taxpayers with very low incomes (such as small part-time earnings of children, or small capital incomes

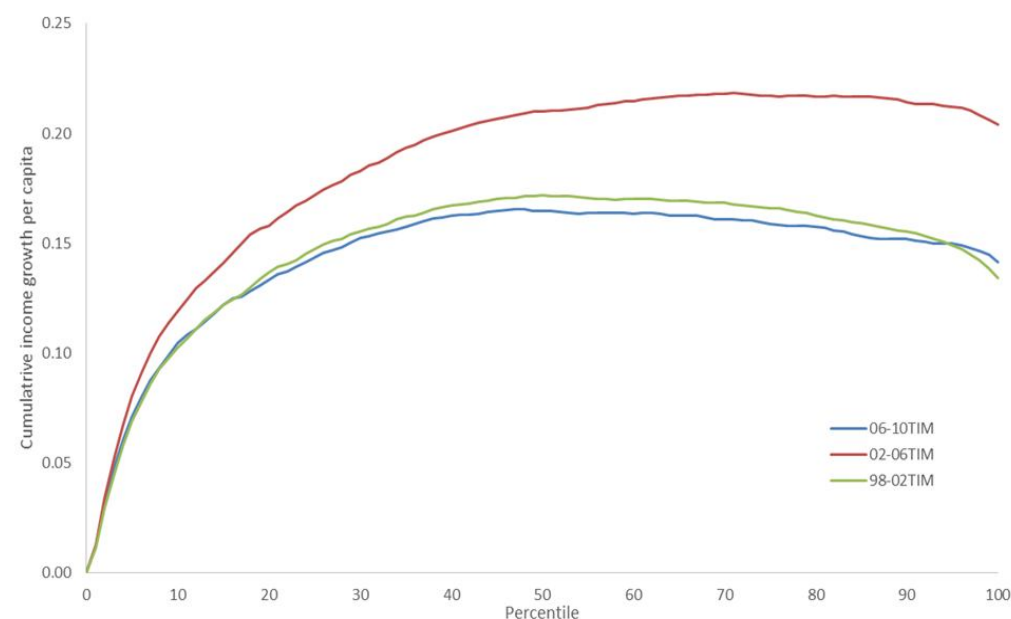


Figure 5: Three TIM Curves for New Zealand

of non-earners), individuals with annual incomes less than \$1,000 were omitted from the sample. This yielded usable samples of 29,405, 31,355 and 32,970 individuals respectively for the three five-year panels. In each case individuals were ranked by their initial year incomes, with all of the diagrams below showing percentiles of the income distribution in the relevant initial year (1998, 2002, or 2006) on the horizontal axis.

5.2 New Zealand TIM Curves

Figure 5 shows three TIM curves corresponding to the three five-year periods (or four years of income growth). Growth rates shown on the vertical axis are measured over the entire period. The right-hand end of the TIM curve represents the average growth rate (over the five years) across all n individuals. While these growth rates were very similar, at around 15% over 1998 to 2002 and 2006 to 2010, it can be seen that growth was higher on average, around 20%, over the period 2002 to 2006.

All three curves tend to rise most steeply at the lowest income percentiles

and flatten out at higher percentiles, suggesting greater equalising mobility especially among the lower percentiles. Also, if higher average income growth across the whole sample (as in the 2002 to 2006 case), is regarded as indicative of higher mobility on average, then mobility is clearly greater across the board in 2002 to 2006 compared to the other two periods. In this case it can be argued that the 2002 to 2006 TIM dominates the other two curves, though the detailed properties of the curves among approximately the lowest 5% are hard to identify on this scale.

If it is preferred to assess mobility from *relative* income growth rates, normalisation of the curves in Figure 5 is required. Figure 6 shows TIM curves normalised by the sample average growth rate in each panel, which also allows the concavity of each curve to be more readily compared. This reveals a quite different dominance pattern with the 2002 to 2006 TIM being dominated by the other two curves. Indeed there would appear to be a clear ranking in dominance terms of 1998 to 2002 > 2006 to 2010 > 2002 to 2006. In addition, while similar patterns are evident in all three curves as h increases, clearly the 2002 to 2006 TIM is less concave, implying less equalising mobility, at any selected h .

5.3 New Zealand Poverty Persistence

Before examining poverty persistence aspects of the data, it should be acknowledged that since these data are based on *individual*, as opposed to *household*, incomes, the notion of a ‘poverty line’ is less meaningful. Clearly many individuals could experience substantial year-to-year changes in income without this necessarily implying that the households of which they are members move into, or out of, poverty. Nevertheless the data serve to illustrate the conceptual aspects of interest.

As mentioned above, identifying poverty persistence first requires a choice of poverty income threshold. For illustrative purposes this is set here at 50% of median income in 2006, where $x_{median} = \$34,087$; $x_p = \$17,044$; hence $h_p = 0.25$ (25%). Figure 7 shows the ‘critical’ growth rate, g^* (the dashed curve) and median actual growth rates within each percentile; there are 329

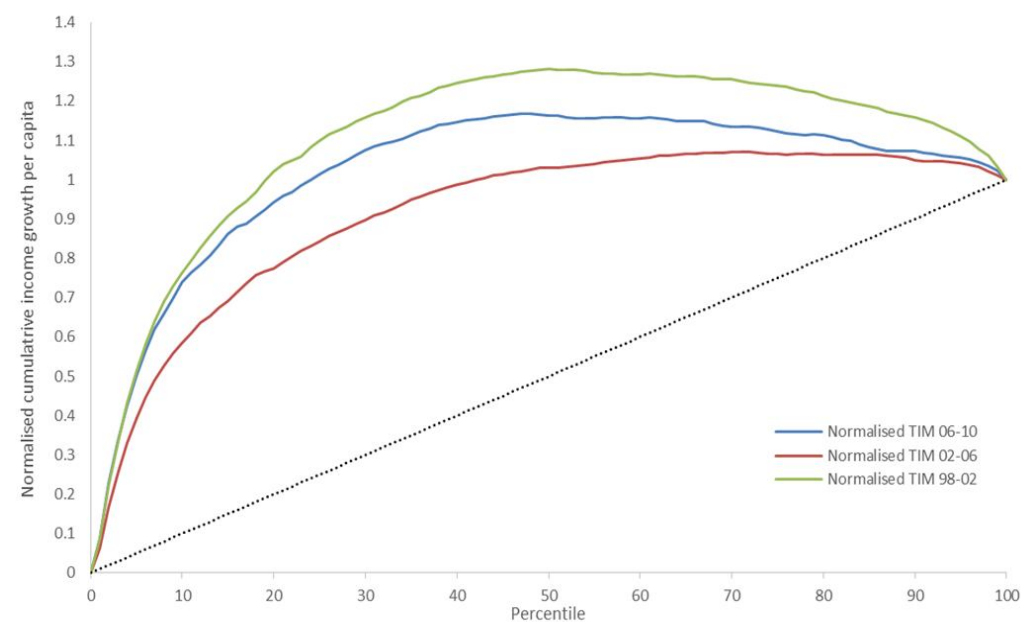


Figure 6: Normalised TIM Curves for New Zealand

individuals in each percentile.

As required, the critical growth rate, g^* , crosses the x -axis at $h_p = 0.25$. To the left of that point, any median growth rate for a given percentile which is above the g^* curve implies that ‘on average’ (i.e. for at least half of the individuals in the percentile) actual income growth was sufficiently large that their income in 2010 exceeded x_p (based on 2006 incomes).

It can be seen from Figure 7 that this condition is satisfied for about 10 of the 25 percentiles below h_p . By contrast, above h_p there are no percentiles for which median growth is sufficiently negative (that is, lying below the g^* curve) to push median individuals below x_p in 2010. This would tend to suggest a high degree of persistence over the 2006 to 2010 period for those not initially in poverty, and somewhat less persistence in poverty for those initially below the poverty line.

However, the median percentile growth rates cannot capture the diversity of experience within each percentile. Figure 8 replaces the percentile medians with box plots for actual percentile growth rates where each ‘box’ shows

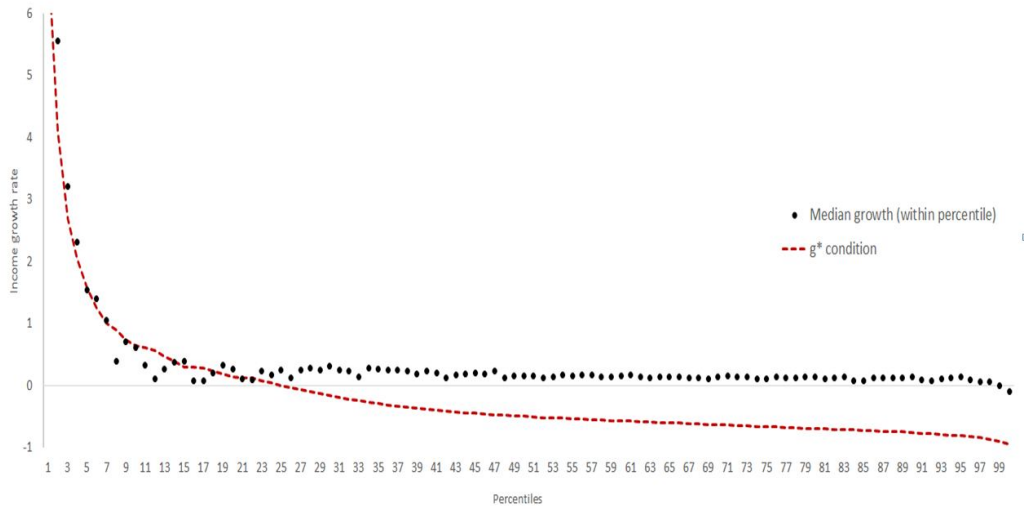


Figure 7: Poverty Persistence in New Zealand

the median growth rate and inter-quartile range. The ‘whiskers’ record the maximum and minimum income growth rate within each percentile.

Figure 8 reveals a quite different pattern of movement into and out of poverty. In particular, the whiskers indicate a wide range of growth rates within each percentile of the initial income distribution, such that, for example, every percentile includes at least one person who moved into poverty. Similarly, for those percentiles initially below $h_p = 0.25$, there is evidence of many individuals in almost all the lower percentiles moving out of poverty; in numerous cases individuals lying between the median and upper quartile (of the percentile distribution) are observed to move out of poverty. This is consistent with a known property of New Zealand personal income taxpayer data; namely the high degree of volatility in individual taxpayer incomes from year to year.

6 Positional Mobility

A widely used class of mobility measures is based on the idea of mobility as ‘positional change’, rather than relative income change. It is therefore useful

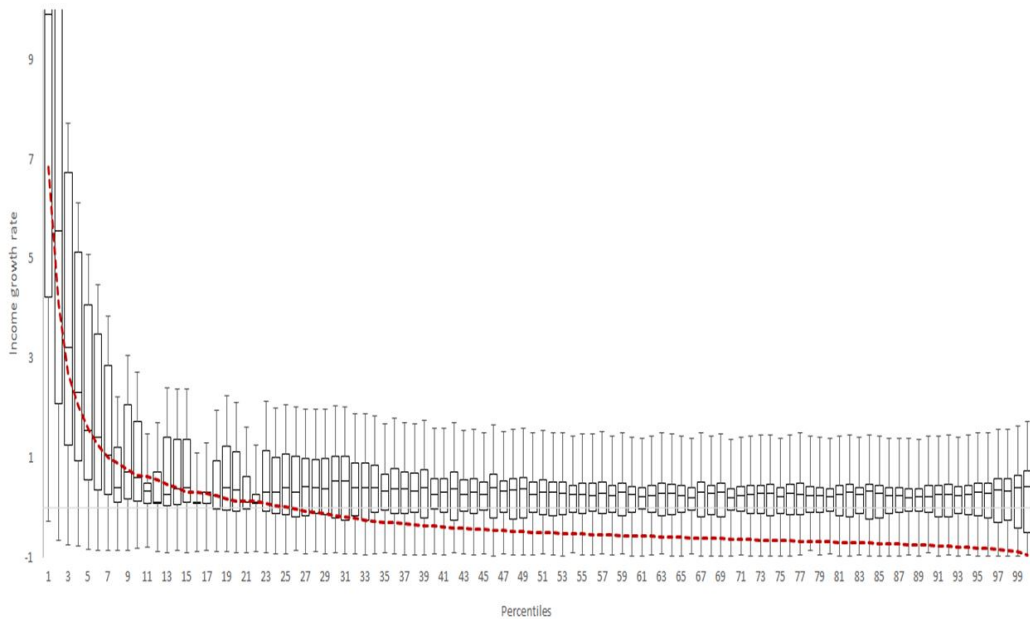


Figure 8: Poverty Persistence in New Zealand: Details

to examine whether the TIM curve approach can be helpful in this context. This section considers re-ranking measures of mobility where changes in individuals' positions in the income distribution, rather than their income levels, are the focus of interest. As is well known, re-ranking measures can be 'directional' or 'non-direction'. These aspects are discussed further below. In the following discussion, individuals are ranked in ascending order of incomes, so that ranks $i = 1, \dots, n$ are for individuals from the lowest to the highest income. If the initial period is denoted 0, then define the initial ranks $R_{i,0} = i$.

Defining a re-ranking mobility index requires, first, a choice regarding *whose* mobility is to be included. Second, it is necessary to decide whether negative re-ranking (dropping down the ranking) is treated symmetrically with positive re-ranking (upward movement within the ranking). Regarding the first choice consider, as in previous sections, the case where it is desired to measure the extent of mobility of a sub-set of individuals, $k \leq n$, with the lowest incomes.

On the treatment of positive and negative re-ranking, let $dR_i = R_{i,1} - R_{i,0} = R_{i,1} - i$ denote the change in the rank order of the person who initially has rank, i . Three further options are possible, all related to how negative re-ranking is treated. Firstly, negative re-ranking could be treated symmetrically with positive re-ranking such that mobility is defined in *net* terms, that is, positive changes in rank net of any negative changes within group $i = 1, \dots, k$.⁸ This is referred to below as ‘*net* re-ranking’ or ‘*net* mobility’. Secondly, negative movement in the ranking can be ignored, which simply involves setting $dR_i = 0$ when $dR_i < 0$. This is referred to below as ‘*gross* re-ranking’ or ‘*gross* mobility’. Thirdly, re-ranking may be measured in *absolute* terms in which all re-ranking is measured as a positive value. This is referred to below as ‘*absolute* re-ranking’ or ‘*absolute* mobility’.

The appropriate choice among these three measures is likely to depend on the question of interest. For example, if interest is focussed on those below y_p as a group, then it may be desired to balance any upward mobility by some of those below y_p with downward (negative) mobility of others below y_p , to gain an indication of the net impact on the group. This would suggest a focus on net mobility in this case. Likewise, if movement *per se* is the mobility concept of interest, then a non-directional measures such as absolute mobility is more relevant.

The three re-ranking mobility indices for an individual initially having rank order, i , (for $i = 1, \dots, n$) can be defined as follows:

$$M_i^{net} = dR_i = R_{i,1} - R_{i,0} \quad (5)$$

$$M_i^{gross} = dR_i|_{dR_i \geq 0} \geq 0 \quad (6)$$

$$dR_i|_{dR_i < 0} = 0.$$

$$M_i^{abs} = |dR_i| = |R_{i,1} - R_{i,0}| \quad (7)$$

Aggregated across the k lowest income individuals in period 0, the corre-

⁸If individual changes in rank are simply aggregated to obtain an aggregate mobility index, then a change in rank of 50 places by one individuals is treated symmetrically as 50 individuals each changing one ranking place.

sponding aggregate mobility indices are then given by:

$$M_k^{net} = \sum_{i=1}^k M_i^{net} = \sum_{i=1}^k (R_{i,1} - R_{i,0}) \quad (8)$$

$$M_k^{gross} = \sum_{i=1}^k M_i^{gross} = \sum_{i=1}^k (R_{i,1} - R_{i,0}) \quad \text{for } dR_i \geq 0 \quad (9)$$

and

$$M_k^{abs} = \sum_{i=1}^k M_i^{abs} = \sum_{i=1}^k |R_{i,1} - R_{i,0}| \quad (10)$$

This last absolute mobility case may be thought of as describing the extent of aggregate movement (positional change) within the relevant range of the income distribution. Over short periods of time this is often described as volatility and is generally regarded as being undesirable. When measured over a longer time period it may be regarded as describing the ‘flexibility’ of the income distribution, with less clear welfare associations.

As above, a TIM curve based on the indices in (8), (9) and (10), plots the cumulative value of the relevant M_k index against the cumulative fraction of the population, $h = k/n$.

Before examining some examples of these re-ranking based TIM curves, it is useful to first consider what maximum mobility would be in each case. Actual mobility may then be compared with these maximum values for any given h .

To simplify the exposition, consider a population of $n = 100$ individuals, each with a different income level: hence each integer represents a percentile of the distribution. They are ranked in period 0, $R_{i,0} = 1 \dots 100$, representing the lowest to the highest incomes. Two polar cases are the maximum and minimum degrees of mobility possible. The former is defined here as a complete ranking reversal, $dR_i(\max)$, such that in period 1, $R_{i,1}$ involves a lowest to highest ranking of $R_{i,1} = 100, \dots, 1$.⁹ Thus $R_{i,0}$ and $R_{i,1}(\max)$ are defined by the sequences in Table 2.

⁹An alternative argument proposes that the relevant comparator should be defined as when the change in an individual’s position in the ranking is purely random; see Jantti and Jenkins (2015; pp.8-9). That is, ‘maximum’ mobility involves independence from initial

Table 2: Maximum Re-ranking

i	1	2	3	...	n
$R_{i,0}$	1	2	3	...	100
$R_{i,1}(max)$	$n + 1 - R_{i,0}$	$n + 1 - R_{i,0}$	$n + 1 - R_{i,0}$...	$n + 1 - R_{i,0}$
	= 100	= 99	= 98	...	= 1

Similarly, the minimum degree of mobility involves no change in the rankings such that $R_{i,1}(\min) = R_{i,0}$ for all i , hence $dR_i = 0$.

From Table 2 it can be seen that maximum mobility implies:

$$M_i(\max) = dR_i(\max) = R_{i,1}(\max) - R_{i,0} = n + 1 - 2R_{i,0} \quad (11)$$

which, for large n , can be approximated by $n - 2R_{i,0}$. Where it is desired to measure the extent of mobility of the sub-set of individuals, $k \leq n$, with the lowest incomes, the aggregate maximum mobility index for the net mobility case, $M_k^{net}(\max)$, is then given by:

$$M_k^{net}(\max) = \sum_{i=1}^k M_i^{net}(\max) = \sum_{i=1}^k (n + 1 - 2R_{i,0}) \quad (12)$$

Further, using the sum of an arithmetic progression:

$$\sum_{i=1}^k R_{i,0} = 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \quad (13)$$

equation (12) becomes:

$$\begin{aligned} M_k^{net}(\max) &= \sum_{i=1}^k (n + 1 - 2R_{i,0}) = k(n + 1) - k(k + 1) \\ &= k(n - k) \end{aligned} \quad (14)$$

Hence, in the $n = 100$ example above, if interest focuses only on the poorest individual ($k = 1$), maximum mobility is given by $M_k^{net}(\max) = (100 -$

positions, rather than complete reversals. In that case, given $R_{i,0}$, maximum mobility requires an actual ordering in period 0 to be compared with a random ordering in period 1. Jantti and Jenkins reject the use of ‘maximum’ when mobility is based on origin independence because, they suggest, “it is difficult to argue that origin independence represents ‘maximum’ mobility in the literal sense”.

1) = 99; when $k = 2$, $M_k^{net}(\max) = 2(100 - 2) = 196$, and so on. More generally, since maximum re-ranking (complete ranking reversal) involves all those below the median individual changing positions with those above the median, it follows from (14) that the maximum value of $M_k^{net}(\max)$ as k increases is obtained for $k = n/2$, yielding $M_k^{net}(\max) = n^2/4$.¹⁰

This measure is therefore not ‘scale independent’: larger populations imply larger $M_k^{net}(\max)$ and M_k^{net} . It could be ‘normalised’ to create a form of per capita index by dividing by n^2 such that the index becomes: $m_k^{net}(\max) = h(1 - h)$. The maximum value is then reached at $h = 0.5$, $m_k^{net}(\max) = 0.25$.

For the gross re-ranking case, $M_k^{gross}(\max)$, the value of $M_k^{gross}(\max)$ also reaches a maximum as k increases of $M_k^{gross}(\max) = n^2/4$ for $k = n/2$, since all individuals below $n/2$ experience positive re-ranking in this (maximum) case. However, above $k = n/2$, as more above-median individuals are included within k , their re-rankings are now given by $dR_i = 0$, such that the cumulative value of $M_k^{gross}(\max)$ remains unchanged as $k \rightarrow n$.

Finally, for the absolute re-ranking case in (10), $M_k^{abs}(\max)$, it can be shown that, as with the other cases, this increases as k increases from $k = 1$ to $k = n/2$ to reach $M_k^{abs}(\max) = n^2/4$. However, this represents a point of inflection rather than a maximum, since inclusion of the absolute value of above-median individuals’ re-ranking in $M_k^{abs}(\max)$, ensures that $M_k^{abs}(\max)$ continues to increase for $k > n/2$, reaching $M_k^{abs}(\max) = n^2/2$ at $k = n$.

7 Re-ranking Measures for New Zealand

7.1 Maximum Re-Ranking profiles

Profiles for the three maximum re-ranking cases discussed above, $M_k^{net}(\max)$, $M_k^{gross}(\max)$, and $M_k^{abs}(\max)$, are illustrated in Figure 9 for the sample of 32,970 New Zealand taxpayers between 2006 and 2010 described in subsection 5.1.

¹⁰Strictly, for small n , the median individual is $k = (n + 1)/2$, and $M_k^{net}(\max)$ is given by $(n + 1)(n - 1)/4$.

This depicts the cumulative percentile of the population on the horizontal axis and the cumulative value of the mobility index on the vertical axis. Hence the maximum value of $M_k^{net}(\max)$ at the 50th percentile is $n^2/4 = 271,155,225$. As noted above these profiles could be ‘normalised’ by dividing all values by n^2 , or by $n^2/4$ to give an index that lies between 0 and 1.

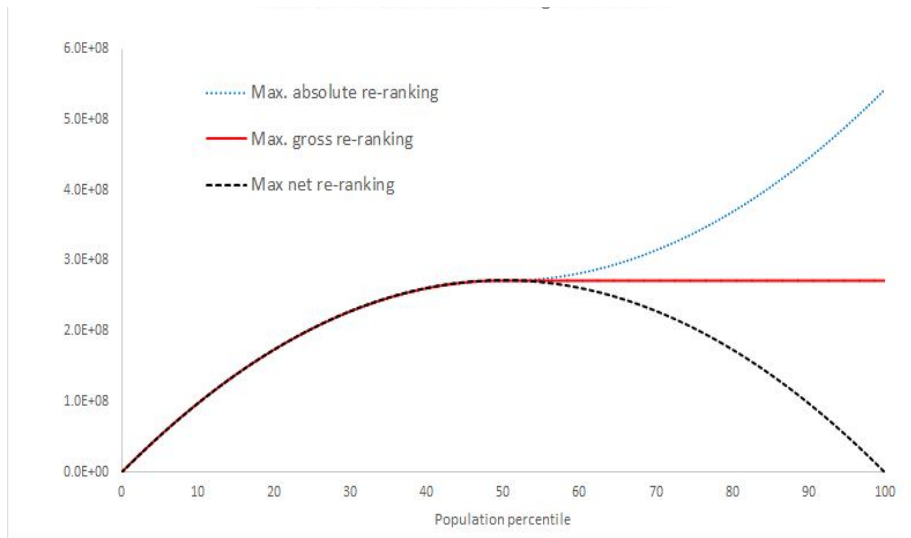


Figure 9: Maximum Re-ranking: Three Cases

Figure 9 shows the distinct non-linear shape of the maximum profiles, whichever definition of mobility is adopted (net, gross or absolute). As expected, the net re-ranking profile displays a parabolic shape which, differentiation of (14) reveals, has a slope of $(n - 2k)$. The equivalent gross re-ranking profile reaches a maximum, as expected, at the 50th percentile and remains constant thereafter, while the absolute re-ranking profile displays a sigmoid shape, reaching a local point of inflection at the 50th percentile but then rising at an increasing rate till the cumulative value of $M_k^{abs}(\max)$ has doubled at $k = n$.

7.2 New Zealand Re-ranking Profiles

An assessment of how much actual mobility occurs, and the roles of incidence, intensity and interpersonal dimensions, is facilitated by plotting equivalents

of the $M_k(\max)$ curves in Figure 9. These are shown in Figure 10, which again shows the non-linear nature of the profiles. In each case, these profiles could contain, concave, linear or convex segments, reflecting the degree of re-ranking being experienced as k is increased to include higher income individuals. A greater amount of re-ranking mobility will tend to generate profiles that are more concave. That is, unlike the poverty TIP curve, greater concavity implies more equalising mobility. Convexity implies disequalising mobility.

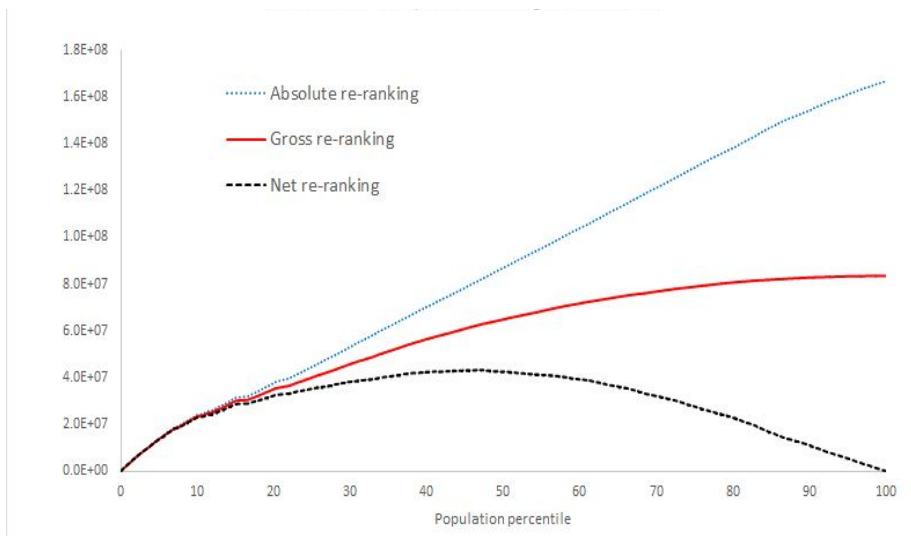


Figure 10: Actual Re-ranking: Three Cases

To assess the incidence, intensity and interpersonal aspects of these mobility measures, Figure 10 should be interpreted as follows. For a given definition of mobility (net, gross or absolute), select a value of $h = k/n$ representing the sub-set of low income individuals of interest (the incidence dimension). The point on the profile at this value of h represents the intensity of mobility for this group; namely how much re-ranking they have experienced on average (or cumulatively). The section of the profile to the right of h becomes irrelevant, equivalent to the flat section of the TIP curve, to the right of p , in the Jenkins-Lambert (1997) analysis.

The deviation from linearity of the M_k profile, from the origin to its value

at the selected h , provides a measure of the degree of equalising (concave) or disequalising (convex) re-ranking within h . That is, the actual profile may be compared to a straight line from the origin to the value of M_k at h . In Figure 10 for example, the profile for absolute re-ranking appears to be remarkably linear, at least above the 10th percentile. This would suggest that, for this sample and measure at least, the extent of mobility via re-ranking of individuals is relatively constant across the income distribution.

An alternative means of determining whether some individuals or groups experience more or less mobility than others is to compare their actual mobility to the maximum mobility achievable. The differences between the actual M_k , and the relevant $M_k(\max)$, curves are shown in the ratios of actual-to-maximum mobility in Figure 11.

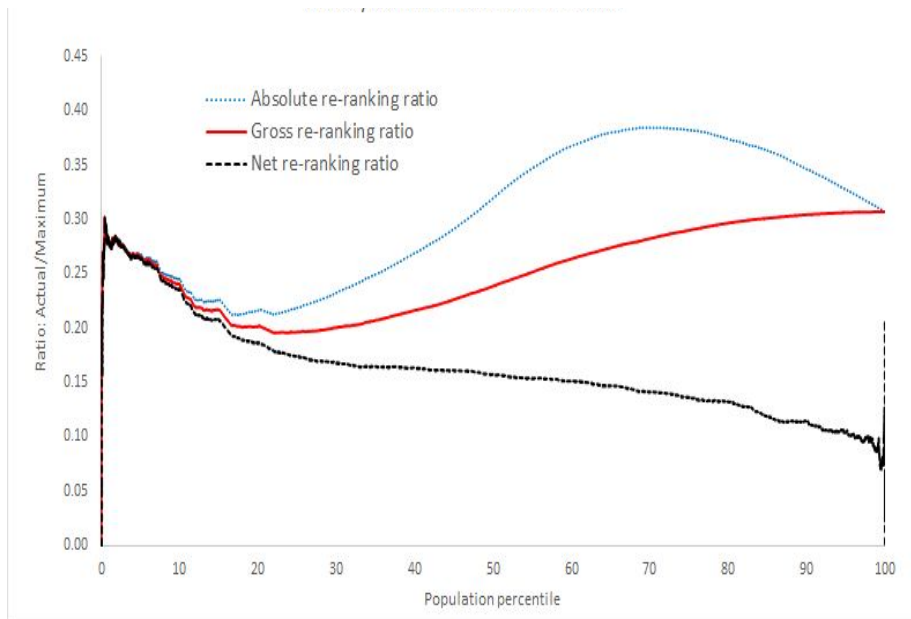


Figure 11: Ratios of Actual to Maximum Re-ranking: Three Cases

This shows that, for all three re-ranking measures, the extent of mobility relative to the maximum achievable, is relatively high for the lowest income individuals (low h), at around 0.25 – 0.3. This steadily declines, as h is increased, to a minimum of approximately 0.2 at around the 20th percentile, except in the case of the M_k^{net} profile which continues to decline for $h > 0.2$,

though at a somewhat slower rate than for $h < 0.2$.¹¹ Thereafter, the ratio rises to around the 70th percentile (and to the 100th percentile in the case of the M_k^{gross} profile). From this it may be inferred that the group experiencing re-ranking that is closest to the maximum achievable are the ‘middle income’ group between approximately the 20th and 70th percentiles.

It can also be seen that the M_k^{gross} and the M_k^{abs} profiles reach the same value for $k = n$. This is not a coincidence. It has already been shown that $M_k^{abs}(\max) = n^2/2$, while $M_k^{gross}(\max) = n^2/4$, at $k = n$; that is, $M_n^{abs}(\max) = 2M_n^{gross}(\max)$. This same relationship holds for the actual measures: $M_n^{abs} = 2M_n^{gross}$. This can be seen by noting that:

$$M_n^{abs} = \sum_1^n |R_{i,1} - R_{i,0}|$$

However, at $k = n$ the sum of positive ranking movements must equal the sum of negative ranking movements, so that:

$$M_n^{abs} = 2 \sum_1^n (R_{i,1} - R_{i,0}) \Big|_{dR_i \geq 0} \quad (15)$$

The term after the summation in (15) is simply the gross re-ranking mobility measure, M_n^{gross} . Hence the actual/maximum ratios for both the gross and absolute re-ranking measures are equal at $k = n$.

Considering the three profiles in Figure 11 it is clear that the measure of net movement, M_k^{net} , indicates a persistent downward trend as h increases towards 1 (100%). This would seem to suggest that the lowest income individuals generally experienced more movement in their incomes (relative to the maximum achievable) over this period than those on higher incomes. It is presumably a re-ranking analogue of the ‘regression to the mean’ in income levels observed above.

8 Conclusions

Almost two decades ago, Jenkins and Lambert (1997) introduced new insights into the poverty measurement literature by demonstrating that various ex-

¹¹The strong fluctuations in the M_H^{net} curve as h approaches 1, reflect the fact that the value of both the actual and maximum net re-ranking measures equal zero at $h = 1$. Hence the ratio can be quite unstable in the vicinity of $h = 1$. (and is, of course, undefined at $h = 1$).

tant measures of the incidence, intensity and inequality of poverty could be integrated and illustrated by their ‘Three "I"s of Poverty’ (TIP) curve. This paper has argued that, despite a wide range of income mobility concepts and measures available in the mobility literature, these three important dimensions of mobility – incidence, intensity and inequality – are also not readily identifiable from current measures or illustrative devices.

However, based on an analogue of the TIP curve, this paper has proposed that a ‘Three "I"s of Mobility’, or TIM, curve can provide a useful means of combining and illustrating these three concepts within a single diagram. For income mobility measured as ‘relative income growth’, this plots the cumulative proportion of the population (from lowest to highest incomes) against the cumulative change in log-incomes per capita over a given period. It was also shown that the analysis can be adapted to illustrate the extent of the poverty persistence component within more general income mobility patterns.

For mobility measures based on the extent of re-ranking of individuals’ incomes over a given period, it was shown that an equivalent curve can illustrate the incidence, intensity and inequality of mobility. This plots the cumulative degree of re-ranking against the cumulative proportion of the population (from lowest to highest incomes). Additionally, since for any given fraction of the population there is a maximum possible extent of re-ranking, it is useful to consider the cumulative ratio of actual-to-maximum re-ranking against the cumulative proportion of the population.

Illustrations for each of these mobility concepts – relative income growth, poverty persistence and re-ranking – were provided based on three panels of New Zealand incomes from 1998 to 2010. These showed that income growth rates within the lower part of the income distribution were quite substantially higher than those observed higher up the income distribution, reflecting in part a relatively high degree of regression towards the mean.

On poverty persistence, average growth rates within each percentile of the distribution suggested relatively little movement into poverty but somewhat more movement out of poverty. However considering all individuals within each percentile (around 330 individuals per percentile in this case) re-

vealed a relatively mobile population overall, with some individuals observed within all percentiles above the (initial year) poverty threshold, moving into poverty over the five year period examined. These figures are of course purely illustrative, depending here on the particular poverty threshold chosen and relating to individual, rather than household, incomes.

Evidence on the extent of re-ranking of individual incomes across a five year period also suggested a relatively high degree of mobility, relative to the maximum possible, among the lowest income individuals and also among those around the 50th to 70th percentiles. It also suggested that the observed extent of re-ranking depends crucially on the re-ranking measure adopted – gross, net or absolute.

Appendix A: Diagram Based on Quantiles of Conditional Distributions

Trede (1998) proposed a graphical illustration of mobility, motivated by the desire to summarise the information contained in a transition matrix. For incomes in t and $t - 1$, the method involved non-parametric estimation of various quantiles of conditional distributions of x_t for given values of x_{t-1} .¹² The quantile profiles are shown in a diagram with income in t on the vertical axis and x_{t-1} on the horizontal axis. Trede suggested translating incomes to relative values by dividing by the mean or median income in each period. A simplified example is shown in Figure 12, which illustrates just three quantiles.

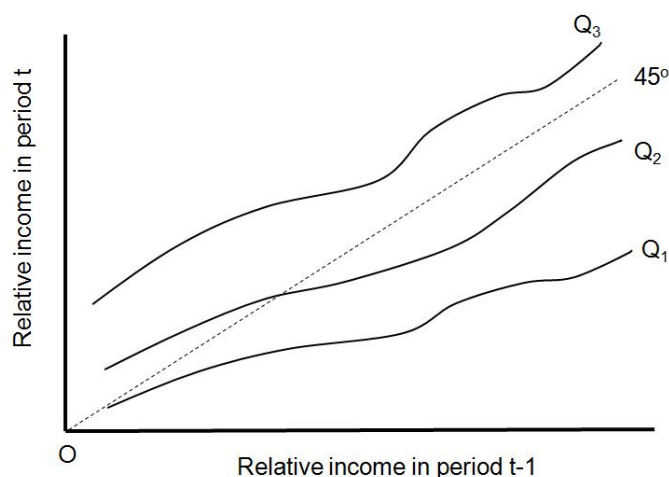


Figure 12: Quantile Regressions of Income in t on Income in $t - 1$

Trede suggests the following interpretation of the quantile profiles. He defined ‘perfect mobility’ as independence of income in $t - 1$, whereby the quantile profiles become horizontal. The vertical distance between quantile profiles give an indication of the extent of income inequality in t .¹³ The

¹²If a sufficient number of observations are available, it would be possible simply to examine the conditional distributions of income in t , for a range of income groups in $t - 1$.

¹³However, the vertical distances refer to conditional distributions, not the marginal distribution in t .

extreme of ‘total [relative] immobility’ produces quantile profiles that coincide: that is, all those with x_{t-1} have the same income in period t . If the (common) quantile profiles coincide with the 45-degree line in the diagram, there is no change in the (marginal) distribution when moving from $t - 1$ to t . Thus, ‘both the distance from each other, and the slopes of the quantile lines provide information about income dynamics’ (Trede, 1998, p. 80).

However, the marginal distributions can remain unchanged even when the quantiles do not coincide and they are not 45-degree lines. For example, consider the simple mobility process in Section 8, and suppose that incomes, x , in two periods are jointly lognormally distributed, so that the $y = \log x$ are jointly normally distributed. Regression towards the (geometric) mean is described by:

$$y_t - \mu_t = \beta (y_{t-1} - \mu_{t-1}) + u_t \quad (\text{A.1})$$

where u_t is a stochastic term with expected value of zero, and variance σ_u^2 . The coefficient, $\beta \leq 1$, indicates the degree to which those below the geometric mean experience, on average, a higher relative income increase than those above the geometric mean. Then:

$$\sigma_t^2 = \beta^2 \sigma_{t-1}^2 + \sigma_u^2 \quad (\text{A.2})$$

There can therefore be no change in the inequality of the marginal distribution of y (so that $\sigma_t^2 = \sigma_{t-1}^2$) if:

$$\beta^2 = 1 - \frac{\sigma_u^2}{\sigma_{t-1}^2} \quad (\text{A.3})$$

Hence, it is not necessary to have $\sigma_u^2 = 0$ and $\beta = 1$ for stability of relative inequality. Furthermore, in this case the conditional distributions of y_t are homoskedastic, so that the quantile profiles are all straight parallel lines. Letting ρ denote the correlation coefficient between log-income in the two periods, it is also known that:

$$\frac{\sigma_t}{\sigma_{t-1}} = \frac{\beta}{\rho} \quad (\text{A.4})$$

Hence stability requires only that the correlation and regression coefficients are equal.

Appendix B: The Growth Incidence Curve and Pro-Poor Growth

The Growth Incidence Curve, introduced by Ravallion and Chen (2001), refers to the change in percentiles of the income (or other welfare metric) distribution from $t-1$ to t ; that is, it is based only on the characteristics of the two relevant cross-sectional distributions. Where $H_t(x)$ is the distribution function of income at t , the p th percentile, $x_t(p)$, is given by:

$$x_t(p) = H_t^{-1}(p) \quad (\text{B.1})$$

The growth rate, $\xi_t(p)$ of the p th percentile is:

$$\xi_t(p) = \frac{x_t(p)}{x_{t-1}(p)} - 1 \quad (\text{B.2})$$

The GIC curve plots $\xi_t(p)$ against p . Since all percentiles are subject to some form of growth, the word ‘incidence’ is perhaps not the most appropriate: $\xi_t(p)$ shows the *extent* of growth of the p th percentile.

Ravallion and Chen (2003) show that $\xi_t(p)$ can be linked to the slopes of the two Lorenz curves. The Lorenz curve is obtained by plotting:

$$L(p) = \frac{1}{\bar{x}} \int_0^{H^{-1}(p)} u dH(u) \quad (\text{B.3})$$

where $p = H(x)$ and \bar{x} is arithmetic mean income, $\int_0^\infty x dH(x)$. The slope, $L'(p)$, is given by:

$$L'(p) = \frac{x(p)}{\bar{x}} \quad (\text{B.4})$$

So that substituting for $x_t(p)$ and $x_{t-1}(p)$ in (B.2) gives:

$$\xi_t(p) = \frac{L'_t(p)}{L'_{t-1}(p)} (\gamma_t + 1) - 1 \quad (\text{B.5})$$

where:

$$\gamma_t = \frac{\bar{x}_t}{\bar{x}_{t-1}} - 1 \quad (\text{B.6})$$

is the growth rate of mean income. Hence if the Lorenz curve is unchanged, $\xi_t(p) = \gamma_t$ for all p : all percentiles grow at the same rate.

Let $p_{x_p, t-1} = \int_0^{x_p} dH_{t-1}(x) = H_{t-1}(x_p)$ denote the headcount poverty measure at $t-1$, where x_p is the constant poverty line: $p_{x_p, t-1}$ is the percentile corresponding to x_p for distribution $H_{t-1}(x)$. Ravallion and Chen (2001) go on to measure the pro-poor growth rate (PPG) by the mean growth rate ‘for the poor’:

$$PPG_t = \frac{1}{p_{x_p, t-1}} \int_0^{p_{x_p, t-1}} \xi_t(p) dp \quad (\text{B.7})$$

Pro-poor growth, defined in this way, leads to a reduction in the Watts (1968) measure of poverty, W_t , defined in terms of a proportional poverty gap and given by:

$$W_t = \frac{1}{p_{x_p, t}} \int_0^{p_{x_p, t}} \log\left(\frac{x_p}{x_t(p)}\right) dp \quad (\text{B.8})$$

Pro-poor growth therefore involves a change in the income distribution that is sufficient to lower the poverty measure. From (B.7) it is clear that the PPG_t measure is directly related to the GIC curve: it is the area under the curve up to, $p_{x_p, t-1}$, the percentile associated with the poverty line.

However, PPG_t is the mean growth rate of percentiles below the fixed poverty line. Clearly, it is not the growth rate of the mean income of those below x_p . Importantly, the wording can lead to misinterpretation, since it is also not the mean growth rate of those individuals who were below x_p in period $t-1$. The GIC is based purely on the two marginal distributions in t and $t-1$: the growth rate of the p th percentile, $\frac{x_t(p)}{x_{t-1}(p)} - 1$, does not refer to the growth rate between t and $t-1$ of the individual at the p th percentile in $t-1$.

Appendix C: An Elasticity Measure of Pro-Poor Growth

Instead of defining pro-poor growth in terms of the arithmetic mean growth rate of percentiles below a fixed poverty line, Essama-Nssah and Lambert (2006) use the concept of the elasticity of poverty with respect to a change in mean income. Letting $D(x)$ denote a ‘deprivation’ measure, with $D(x) = 0$ for $x \geq x_p$ and $D(x)$ is a decreasing convex function of x for $x < x_p$, they

consider the set of ‘average deprivation’ poverty measures defined by:

$$P = \int_0^{x_p} D(x) dH(x) \quad (\text{C.1})$$

Define $\eta_{P,\bar{x}}$ as the elasticity of P with respect to changes in arithmetic mean income, \bar{x} . Then:

$$\eta_{P,\bar{x}} = \frac{\bar{x}}{P} \frac{dP}{d\bar{x}} = \frac{\bar{x}}{P} \int_0^{x_p} \frac{dD(x)}{dx} \frac{dx}{d\bar{x}} dH(x) \quad (\text{C.2})$$

Define $\eta_{x,\bar{x}}$ as the elasticity of x with respect to changes in mean income. The $\eta_{x,\bar{x}}$ s describe the individual income dynamics over the period. Then (C.2) can be written, using $D'(x) = \frac{dD(x)}{dx}$, as:¹⁴

$$\eta_{P,\bar{x}} = \frac{1}{P} \int_0^{x_p} x D'(x) \eta_{x,\bar{x}} dH(x) \quad (\text{C.3})$$

It is important to recognise that Essama-Nssah and Lambert (2006) implicitly assume that the individual income changes are such that there are no movements across the poverty line, x_p . For the headcount poverty measure, which requires $D(x) = 1$ for $x < x_p$, equation (C.3) gives $\eta_{P,\bar{x}} = 0$. However, consider the simple case where $D(x) = 1 - x/x_p$, for which:

$$P = H(x_p) \left(1 - \frac{\bar{x}_p}{x_p}\right) \quad (\text{C.4})$$

where, as above, $H(x_p)$ is the headcount poverty measure and \bar{x}_p is the arithmetic mean income of those below the poverty line. Using $D'(x) = -\frac{1}{x_p}$, the elasticity becomes:

$$\eta_{P,\bar{x}} = -\frac{1}{H(x_p)(x_p - \bar{x}_p)} \int_0^{x_p} x \eta_{x,\bar{x}} dH(x) \quad (\text{C.5})$$

Define \hat{x}_p as the weighted average income of those in poverty, with weights $\eta_{x,\bar{x}}$. Hence $\hat{x}_p = \frac{1}{H(x_p)} \int_0^{x_p} x \eta_{x,\bar{x}} dH(x)$ and:

$$\eta_{P,\bar{x}} = -\frac{\hat{x}_p}{x_p - \bar{x}_p} \quad (\text{C.6})$$

¹⁴Is it possible to introduce another elasticity $\eta_{D,x}$, so that $\eta_{P,\bar{x}} = \int_0^{x_p} D'(x) \eta_{D,x} \eta_{x,\bar{x}} dH(x)$.

The elasticity, $\eta_{P,\bar{x}}$, can be rewritten as:

$$\eta_{P,\bar{x}} = \frac{1}{P} \int_0^{x_p} x D'(x) dH(x) + \frac{1}{P} \int_0^{x_p} x D'(x) (\eta_{x,\bar{x}} - 1) dH(x) \quad (\text{C.7})$$

The first term in (C.7) is the elasticity that would result from uniform income growth of $\eta_{x,\bar{x}} = 1$ for all x . The second term reflects the contribution of the deviation from uniform growth. Essama-Nssah and Lambert (2006) define the extent of pro-poor growth, π , as:

$$\pi = P \left(\eta_{P,\bar{x}} \Big|_{\eta_{x,\bar{x}}=1} - \eta_{P,\bar{x}} \right) \quad (\text{C.8})$$

where $\eta_{P,\bar{x}} \Big|_{\eta_{x,\bar{x}}=1}$ is the elasticity resulting from uniform income growth, that is, the first term in (C.7). Hence:

$$\pi = - \int_0^{x_p} x D'(x) (\eta_{x,\bar{x}} - 1) dH(x) \quad (\text{C.9})$$

For example, in the above case where $D(x) = 1 - x/x_p$, it can be seen that:

$$\pi = -H(x_p) \left(\frac{\hat{x}_p - \bar{x}_p}{x_p} \right) \quad (\text{C.10})$$

In the case where the poorest of the poor experience relatively larger (and positive) income changes compared with those closer to the poverty line, $\hat{x}_p < \bar{x}_p$ and π is positive.

Appendix D: Movements Across the Poverty Line

For an individual with initial income of x , experiencing a proportional change of dx/x , poverty is avoided if:

$$x \left(1 + \frac{dx}{x} \right) > x_p \quad (\text{D.1})$$

Hence the income change must be such that:

$$\frac{dx}{x} > \frac{x_p}{x} - 1 \quad (\text{D.2})$$

Suppose the dynamics can be described by the following function:

$$\frac{dx}{x} = g + f(x) + u \quad (\text{D.3})$$

where g denotes the general growth in incomes, u is a stochastic term distributed as $N(0, \sigma^2)$, and the function $f(x)$ describes the relative income movements. For example, for the basic regression towards the mean model, $f(x) = -(1 - \beta) \log(x/G)$, where β is the regression coefficient and G is the geometric mean income in the initial period. Hence to avoid poverty, it is required that:

$$g + f(x) + u > \frac{x_p}{x} - 1 \quad (\text{D.4})$$

or:

$$u > \frac{x_p}{x} - (1 + g) - f(x) = s(x, x_p, g) \quad (\text{D.5})$$

The probability of avoiding poverty is thus:

$$\int_{s(x, x_p, g)}^{\infty} dN(u|0, \sigma^2) = 1 - \int_{-\infty}^{s(x, x_p, g)} dN(u|0, \sigma^2) \quad (\text{D.6})$$

The probability of being in poverty in the second period is:

$$N\left(\frac{s(x, x_p, g)}{\sigma} \middle| 0, 1\right) \quad (\text{D.7})$$

Where, as above, $H(x_p)$ is the initial headcount poverty measure, the headcount poverty measure in the second period is a weighted average of individual probabilities, given by:

$$\int_0^{\infty} N\left(\frac{s(x, x_p, g)}{\sigma} \middle| 0, 1\right) dH(x) \quad (\text{D.8})$$

Appendix E: The TIM Curve and Regression to the Mean

Consider the properties of the TIM curve for the case of a specific process of relative income change. Regression towards the (geometric) mean is described by:

$$y_t - \mu_t = \beta(y_{t-1} - \mu_{t-1}) + u_t \quad (\text{E.1})$$

where u_t is a stochastic term with expected value of zero, and $\beta \leq 1$ indicates the degree to which those below the geometric mean experience, on average, a higher relative income increase than those above the geometric mean.¹⁵

Substituting for $y_t - \mu_t$ from (E.1) into (3) gives:

$$\begin{aligned} M_{h,t} = & \left[(\beta - 1) \int_0^{y_{h,t-1}} (y_{t-1} - \mu_{t-1}) dF(y_{t-1}) \right] \\ & + [(\mu_t - \mu_{t-1}) F(y_{h,t-1})] \\ & + \left[\int_0^{y_{h,t-1}} u_t dF(y_{t-1}) \right] \end{aligned} \quad (\text{E.2})$$

The ‘height’ of the TIM curve, at any value of h , is thus made up of three components, each contained within square brackets. The first term is $(\beta - 1)$ multiplied by the sum, up to $y_{h,t-1} = F^{-1}(h)$, of the differences between log-income and mean log-income in period $t - 1$ (or the sum of the logarithms of relative income, x_t/G_t). The second term is h multiplied by the overall growth rate of (geometric mean) income: this term has a linear profile. The third term is the sum of the stochastic terms. For values of $y_{t-1} < \mu_{t-1}$ (incomes below the geometric mean), the slope of $M_{h,t}$ is positive. A turning point occurs for $y_{h,t-1} = \mu_{t-1}$, after which the slope is negative (since $y_{t-1} > \mu_{t-1}$ and $\beta < 1$).

Consider the component of the TIM curve, $M_{h,t}^R$, say, that reflects only the systematic component of relative income changes, the regression towards the mean. Then:

$$M_{h,t}^R = (\beta - 1) \int_0^{y_{h,t-1}} (y_{t-1} - \mu_{t-1}) dF(y_{t-1}) \quad (\text{E.3})$$

Furthermore, let $F_1(y)$ denote the first moment distribution function of log-income, the proportion of total log-income obtained by those with log-income below y . Hence a graph of $F_1(y)$ plotted against $F(y)$ gives the Lorenz curve of log-income. Then:

$$M_{h,t}^R = (\beta - 1) \mu_{t-1} \{F_1(y_{h,t-1}) - F(y_{h,t-1})\} \quad (\text{E.4})$$

¹⁵For any individual, income in period t is given by: $Y_t = (Y_{t-1}/G_{t-1})^\beta \exp\{(\mu_t - \mu_{t-1}) + u_t\}$.

Differentiating:

$$\frac{dM_{h,t}^R}{dF(y_{h,t-1})} = (\beta - 1) \mu_{t-1} \left(\frac{dF_1(y_{h,t-1})}{dF(y_{h,t-1})} - 1 \right) \quad (\text{E.5})$$

and:

$$\frac{d^2 M_{h,t}^R}{dF(y_{h,t-1})^2} = (\beta - 1) \mu_{t-1} \frac{d^2 F_1(y_{h,t-1})}{dF(y_{h,t-1})^2} \quad (\text{E.6})$$

Simplifying notation using, for example, $\frac{dF_1(y_{h,t-1})}{dF(y_{h,t-1})} = \frac{dF_1}{dF}$, the concavity, C , of M_h^R at any given h is expressed as:

$$C = \frac{\frac{d^2 F_1}{dF^2}}{1 - \frac{dF_1}{dF}} F \quad (\text{E.7})$$

This is independent of β .

While the M_h^R is smooth, the concavity at any point does not depend on the regression coefficient, β . However, more regression (a lower value of β) means that the profile deviates further from a straight line, and lies everywhere above the profile obtained from a higher β . For example, the maximum height of this component of the TIM curve is equal to:

$$(1 - \beta) \mu_{t-1} \{ F_1(\mu_{t-1}) - F(\mu_{t-1}) \} \quad (\text{E.8})$$

and this clearly depends on β . The term in curly brackets is the maximum vertical distance between the Lorenz curve of log-income and the diagonal of equality.

From (E.5), the slope of M_h^R at any point does depend on β . It also depends on the slope of the Lorenz curve of log-income at the corresponding value of $h = F(y_{h,t})$. Up to the arithmetic mean of log-income, the slope $dF_1/dF < 1$ and above the mean the slope is greater than 1.

The slope of a ray from the origin to a point on the M_h^R component of the TIM curve is:

$$(\beta - 1) \mu_{t-1} \left(\frac{F_1}{F} - 1 \right) \quad (\text{E.9})$$

and this of course is always positive. This slope depends on regression, and on the slope of a ray from the origin to the corresponding point on the Lorenz curve of log-income in $t - 1$.

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About the Authors

John Creedy is Professor of Public Economics and Taxation at Victoria Business School, Victoria University of Wellington, New Zealand, and a Principal Advisor at the New Zealand Treasury.

Email: john.creedy@vuw.ac.nz

Norman Gemmell is Professor of Public Finance at Victoria Business School, Victoria University of Wellington, New Zealand.

Email: norman.gemmell@vuw.ac.nz

