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Nonexistence of exact solutions agreeing with the Gaussian beam on the beam axis or in the focal plane

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A R T I C L E I N F O

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A B S T R A C T

Solutions of the Helmholtz equation which describe electromagnetic beams (and also acoustic or particle beams) are discussed. We show that *an exact solution which reproduces the Gaussian beam waveform on the beam axis does not exist*. This is surprising, since the Gaussian beam is a solution of the paraxial equation, and thus supposedly accurate on and near the beam axis. Likewise, a solution of the Helmholtz equation which exactly reproduces the Gaussian beam in the focal plane does not exist. We show that the last statement also holds for Bessel–Gauss beams. However, solutions of the Helmholtz equation (one of which is discussed in detail) *can* approximate the Gaussian waveform within the central focal region.

1. Introduction

Acoustic, electromagnetic and particle beams are described by solutions of the Helmholtz equation

$$\left(\nabla^2 + k^2\right)\psi = 0. \tag{1}$$

Eq. (1) results respectively from the linearized hydrodynamic equations, the Maxwell equations, and the Schrödinger equation. The wavenumber $k = \omega/c$ (with ω the angular frequency and c the speed of sound or the speed of light), or is related to the energy per particle, which for the particle mass M is $\hbar^2 k^2/2M$.

The widely used but approximate solution known as the Gaussian beam ([1], Section 16.7, [2], Section 20.3) is

$$\psi_G(x, y, z) = \frac{b}{b+iz}\exp\left\{ikz - \frac{k\left(x^2+y^2\right)}{2(b+iz)}\right\} \tag{2}$$

ψ_G is the fundamental mode solution of the *paraxial equation*, obtained by setting $\psi = e^{ikz}G$ in the Helmholtz equation and then neglecting the term $\partial_z^2 G$ in the resulting equation for G (given below). This amounts to assuming that the dominant z-dependence of the beam lies in the e^{ikz} factor (when propagation is in the z direction, as is assumed here). For axially symmetric solutions we omit the azimuthal derivative, so the Helmholtz equation in cylindrical coordinates, with $\rho = \sqrt{x^2+y^2}$ the distance from the beam axis, takes the form

$$\left(\partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2 + k^2\right)\psi = 0. \tag{3}$$

The substitution $\psi = e^{ikz}G$ gives an equation for G, namely $(\partial_\rho^2 + \rho^{-1}\partial_\rho + 2ik\partial_z + \partial_z^2)G = 0$, in paraxial form $(\partial_\rho^2 + \rho^{-1}\partial_\rho + 2ik\partial_z)G \approx 0$. This paraxial equation has as fundamental solution $G = \frac{b}{b+iz}\exp\left\{-\frac{k\rho^2}{2(b+iz)}\right\}$, thus giving us ψ_G of Eq. (2). Higher modes may be obtained by differentiation of ψ_G with respect to x, y or z, since the differential equations are unchanged by translation in any coordinate. ψ_G depends on the wavenumber k and on the length b, which gives the longitudinal extent of the focal region. The transverse extent in the focal plane is given by $w_0 = \sqrt{2b/k}$. Thus the Gaussian fundamental mode is characterized by a single dimensionless parameter kb. It may seem plausible that when $kb \gg 1$ (focal region large longitudinally compared to k^{-1}) the Gaussian beam would become a satisfactory solution of the Helmholtz equation, everywhere. This is not so: when ψ_G is substituted into $k^{-2}\psi_G^{-1}$ times the Helmholtz equation, we obtain ([2], Section 20.3), instead of zero, $\frac{2}{k^2(b+iz)^2} - \frac{2\rho^2}{k(b+iz)^3} + \frac{\rho^4}{4(b+iz)^4}$. It follows that the errors become small in regions where *both* of the following inequalities hold:

$$k^2\left(b^2+z^2\right) \gg 1 \quad \text{and} \quad b^2+z^2 \gg \rho^2 \tag{4}$$

Fig. 1 shows the modulus and phase of ψ_G, for $kb = 2$. Since the exponent of the modulus of ψ_G tends to $-kb\rho^2/2z^2 = -(kb/2)\tan^2\theta$ far from the origin, the half-angle of the cone of divergence of the Gaussian beam, obtained by setting the exponent equal to -1, is $\theta = \arctan\sqrt{2/kb}$. The divergence angle defined in this way is 45° for $kb = 2$, and 30° for $kb = 6$.

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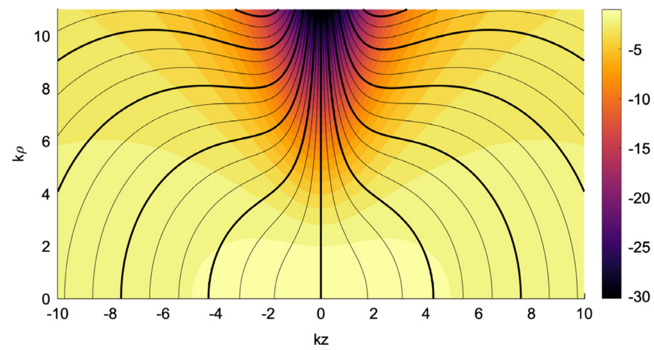

Fig. 1. $\psi_G(\rho, z)$ in the focal region, plotted for $kb = 2$, for $k|z| \leq 10, k\rho \leq 10$. Shading indicates modulus of the wavefunction (logarithmic scale, lighter colour indicates larger modulus). The isophase surfaces are shown at intervals of $\pi/3$. The phase is chosen to be zero at the origin. The isophase contours that are multiples of π are drawn with heavier lines. The three-dimensional picture is obtained by rotating the figure about the beam axis (the horizontal axis). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

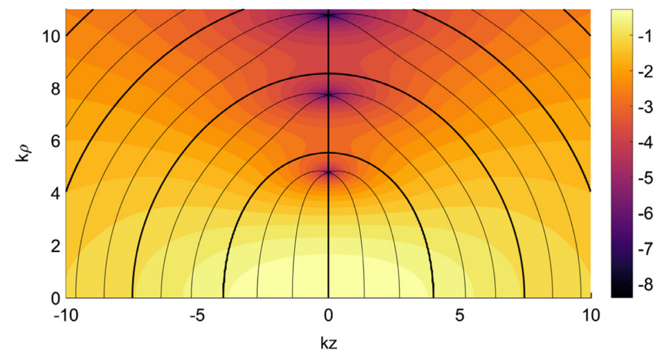

Fig. 2. $\psi_b(\rho, z)$ in the focal region, plotted for $kb = 2$, for $k|z| \leq 10, k\rho \leq 10$. Shading indicates modulus of the wavefunction (logarithmic scale, lighter colour indicates larger modulus). The isophase surfaces are shown at intervals of $\pi/3$. The phase is chosen to be zero at the origin. The isophase contours, other than those that are multiples of π, meet on the zeros of $\psi_0(\rho, z)$, three of which are shown, at $k\rho \approx 4.77, 7.73$ and 10.77. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Many authors [3–10] have investigated methods to build up exact solutions of the Helmholtz equation from solutions of the paraxial equation, typically as expansions in powers of $(kb)^{-1}$ or of $w_0/b = \sqrt{2/kb}$. These expansions have problems, not just in complexity, but in boundedness as well. A case in point is Wünsche's [6] operator method, which aims to get exact solutions from paraxial solutions by acting on the latter with differential operators (given as infinite series of partial derivatives with respect to z). We touch on this method in Appendix C.

Our aim here is different: we ask the question 'can *any* physical solution of the Helmholtz equation duplicate the Gaussian beam on the axis, or in the focal plane?' The answer to both questions is 'no', such solutions do not exist. (By 'physical' is meant causal and having finite beam invariants, as explained in the next Section.) This will be shown in Sections 4 and 5. We also show, in Appendix B, that no Bessel–Gauss beam can be the same in its focal plane as an exact solution. But first we compare and contrast a recent exact solution with the Gaussian beam, in Sections 2 and 3.

2. An exact solution and its properties

A recent paper [11] discusses solutions of the Helmholtz equation (1) which represent transversely bounded beams, of the form

$$\psi(\boldsymbol{r}) = e^{im\phi} \int_0^k dq f(k,q) J_m\left(\rho\sqrt{k^2-q^2}\right) e^{iqz}. \tag{5}$$

Beams of this form propagate along the z direction. The wave motion is causal [11], meaning that far from the focal region there is no backward propagation. Ref. [11] discusses wavefunctions with no azimuthal dependence ($m = 0$), and gives an explicit expression for the case where $f(k,q)$ is proportional to q, in terms of Lommel functions of two variables, or equivalently in terms of products of spherical Bessel and Legendre functions. Proportionality to q at small q is sufficient to ensure the finiteness of beam invariants and of physical quantities such as the energy content per unit length of the beam ([11], Sections 5 and 6).

Wavefunctions of the form (5) can all be generalized to

$$\psi(\boldsymbol{r}) = e^{im\phi} \int_0^k dq f(k,q) J_m\left(\rho\sqrt{k^2-q^2}\right) e^{iq(z-ib)}. \tag{6}$$

The imaginary translation in z, which leads to the extra factor e^{qb} in the integrand, leaves the Laplacian unchanged, so (6) is still an exact solution of (1).

The $m = 0$ beam with $f(k,q)$ set equal to a constant in (6) does approximate ψ_G uniformly along the beam axis, with error of order e^{-kb}. However, $f(k,q) =$ constant is not physically possible: it does not give a finite energy content per unit length of the beam. For example, the corresponding transverse magnetic (TM) beam has energy content per unit length of the beam ([12], Eq. (26)) proportional to $\int_0^k dq q^{-1}\left(k^2-q^2\right) e^{2qb}$, which diverges logarithmically.

As an example of a beam waveform which has all of the required physical properties, we shall consider the wavefunction of Section 9 of [11]:

$$\psi_b(\rho,z) = \frac{b^2}{\left[e^{kb}(kb-1)+1\right]} \int_0^k dq\, q\, e^{q(b+iz)} J_0\left(\rho\sqrt{k^2-q^2}\right). \tag{7}$$

The prefactor in (7) normalizes the wavefunction to unity at the origin $\rho = 0, z = 0$, for easier comparison with ψ_G given in (2), which is also normalized to unity at the origin.

We shall show that, for $e^{kb} \gg 1$ and $\rho^2 \ll b/k$, where the Gaussian waveform has some validity, the wavefunction ψ_b corresponds closely to it, provided that also $|z| \ll kb^2$. There are no constraints on where ψ_b may be used, being an exact solution of (1). Fig. 2 shows $\psi_b(\rho, z)$ in the focal region around the origin, for kb again set equal to 2.

On the beam axis $\rho = 0$ we have

$$\begin{aligned}\psi_b(0,z) &= \frac{b^2}{\left[e^{kb}(kb-1)+1\right]} \int_0^k dq\, q\, e^{q(b+iz)} \\ &= \left(\frac{b}{b+iz}\right)^2 \frac{e^{k(b+iz)}\left[k(b+iz)-1\right]+1}{\left[e^{kb}(kb-1)+1\right]}.\end{aligned} \tag{8}$$

An explicit form of ψ_b at a general point (ρ, z) was found in [11], using the fact that the expression (7) is a cylindrically symmetric non-singular solution of the Helmholtz equation, and may thus be expanded as a sum over products of Legendre polynomials and spherical Bessels,

$$\begin{aligned}\psi_b(\rho,z) &= \frac{(kb)^2}{\left[e^{kb}(kb-1)+1\right]} \sum a_n P_n\left(\frac{z-ib}{R}\right) j_n(kR), \\ R &= (z-ib)\sqrt{1+\rho^2/(z-ib)^2}.\end{aligned} \tag{9}$$

As in Ref. [11], R is chosen as a branch of the complex radial coordinate resulting from an imaginary displacement along the beam axis:

$$r = \sqrt{\rho^2+z^2} \to R = \sqrt{\rho^2+(z-ib)^2}. \tag{10}$$

The coefficients a_n in the expansion are given in [11], Appendix B. There is only one non-zero odd coefficient, $a_1 = 2i$. The even coefficients we shall rename as $a_{2n} = A_n$, so that

$$\begin{aligned}\psi_b(\rho,z) = \frac{(kb)^2}{\left[e^{kb}(kb-1)+1\right]} &\left\{2iP_1\left(\frac{z-ib}{R}\right) j_1(kR)\right. \\ &\left. + \sum_0^\infty A_n P_{2n}\left(\frac{z-ib}{R}\right) j_{2n}(kR)\right\}.\end{aligned} \tag{11}$$

The full sequence of the coefficients of the even terms is (note that $(-1)!! = 1$)

$$A_0 = 1, \qquad A_n = -\frac{(4n+1)(2n-3)!!}{2^n (n+1)!} = -\frac{2(4n+1)(2n-3)!!}{(2n+2)!!}, \quad n = 1,2,3,\ldots \tag{12}$$

At the origin $\rho = 0, z = 0$ we have $R \to -ib$, $P_n\left(\frac{z-ib}{R}\right) \to P_n(1) = 1$, so (we set $kb = \beta$)

$$\psi_b(0,0) = \frac{\beta^2}{\left[e^{\beta}(\beta-1)+1\right]}\{2ij_1(-i\beta) + \sum_0^{\infty} A_n j_{2n}(-i\beta)\}. \tag{13}$$

The right-hand side is unity because of the identities given in Eq.(9.6) of [11].

In the focal plane $z = 0$ we have $R \to -i\sqrt{b^2-\rho^2}$, so

$$\psi_b(\rho,0) = \frac{\beta^2}{\left[e^{\beta}(\beta-1)+1\right]}\left\{2iP_1\left(\frac{b}{\sqrt{b^2-\rho^2}}\right) j_1\left(-ik\sqrt{b^2-\rho^2}\right) + \sum_{n=0}^{\infty} A_n P_{2n}\left(\frac{b}{\sqrt{b^2-\rho^2}}\right) j_{2n}\left(-ik\sqrt{b^2-\rho^2}\right)\right\}. \tag{14}$$

There is a removable singularity on the circle $\rho = b$ in the $z = 0$ plane, since for large X and small x

$$P_n(X) \sim \frac{(2n-1)!!}{n!}X^n, \qquad j_n(x) \sim \frac{x^n}{(2n+1)!!}. \tag{15}$$

Thus as $\rho \to b$ the quantity in braces in (14) tends to

$$1 + \frac{2\beta}{3} + \sum_{n=1}^{\infty} A_n \frac{\left(-\beta^2\right)^n}{(4n+1)(2n)!} = 1 + \frac{2\beta}{3} - \sum_{n=1}^{\infty} \frac{\left(-\beta^2/4\right)^n}{n!(n+1)!(2n-1)}. \tag{16}$$

The final sum may be evaluated as

$$1 - \frac{2}{3}\left[J_0(\beta) + \beta^{-1}J_1(\beta)\right] - \frac{2\beta^2}{3}\left[J_0(\beta) - \beta^{-1}J_1(\beta)\right] + \frac{\pi\beta^2}{3}\left[J_0(\beta)H_1(\beta) - J_1(\beta)H_0(\beta)\right]. \tag{17}$$

In this expression J_0, J_1 are the usual Bessel functions, and H_0, H_1 are the related Struve functions (Chapters 10 and 11 of [13]).

The focal plane wavefunction (14) is real for all ρ. It becomes oscillatory for $\rho > b$, in which case

$$\beta^{-2}\left[e^{\beta}(\beta-1)+1\right]\psi_b(\rho,0) = \frac{2b}{\sqrt{\rho^2-b^2}} j_1\left(\sqrt{\rho^2-b^2}\right) + \sum_{n=0}^{\infty} A_n P_{2n}\left(\frac{-ib}{\sqrt{\rho^2-b^2}}\right) j_{2n}\left(k\sqrt{\rho^2-b^2}\right). \tag{18}$$

When $\rho^2 \gg b^2$ the identity (B1) of [11] shows that the right-hand side of (18) tends to $2J_1(k\rho)/k\rho$. As expected, there is an infinite number of circles of wavefunction zeros in the focal plane, at which the different isophase surfaces from opposite sides of the focal plane can meet ([2], Section 20.1.4).

The divergence angle of the beam is found in Appendix A. For large kb it is the same as for the Gaussian beam, $\theta \approx \sqrt{2/kb}$.

3. The ratio of $\psi_G(\rho,z)$ to $\psi_b(\rho,z)$

Both $\psi_G(\rho,z)$ and $\psi_b(\rho,z)$ are normalized to unity at the origin, chosen as the centre of the focal region. On the beam axis we have, and with $\beta = kb, \zeta = kz$,

$$\psi_G(\rho,z) = \left(\frac{\beta}{\beta+i\zeta}\right)e^{i\zeta},$$
$$\psi_b(0,z) = \left(\frac{\beta}{\beta+i\zeta}\right)^2 \frac{\left[(\beta+i\zeta-1)e^{\beta+i\zeta}+1\right]}{(\beta-1)e^{\beta}+1}$$
$$\frac{\psi_G(0,z)}{\psi_b(0,z)} = \frac{(\beta+i\zeta)e^{i\zeta}\left[(\beta-1)e^{\beta}+1\right]}{\beta\left[(\beta+i\zeta-1)e^{\beta+i\zeta}+1\right]}. \tag{19}$$

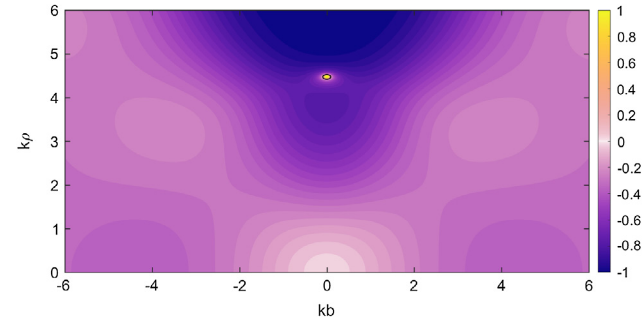

Fig. 3. Ratio of the moduli $\left|\psi_G(\rho,z)/\psi_b(\rho,z)\right|$ in the focal region, plotted for $kb = 2$. Shading indicates modulus of the ratio (colour bar indicates value of modulus ratio minus unity, so zero corresponds to the best agreement). The ratio tends to infinity at the zeros of $\psi_b(\rho,0)$, one of which is shown, at $k\rho \approx 4.77$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Thus the ratio of ψ_G to ψ_b on the axis is, for $\beta \gg 1$,

$$\frac{\psi_G(0,z)}{\psi_b(0,z)} = \frac{(\beta+i\zeta)(\beta-1)}{\beta(\beta+i\zeta-1)} + O\left(e^{-\beta}\right) = 1 - \frac{i\zeta}{\beta^2} + O\left(\beta^{-3}\right) + O\left(e^{-\beta}\right). \tag{20}$$

It follows that the ratio of the beam wavefunctions is close to unity when $\beta \gg 1$ and also $|z| \ll kb^2$.

We do not expect close correspondence far from the origin in the focal plane $z = 0$, since the Gaussian beam has no zeros in the focal plane, while $\psi_b(\rho,0)$ has an infinity of zeros, as we saw in the previous Section. Expansion of (14) in powers of ρ gives

$$\frac{\psi_G(\rho,0)}{\psi_b(\rho,0)} = 1 + \frac{1}{4}\frac{\left(2\beta^2-4\beta+4\right)+\beta^2-4}{e^{\beta}(\beta-1)+1}\frac{\rho^2}{b^2} + O\left(\frac{\rho^4}{b^4}\right). \tag{21}$$

For $kb = \beta \gg 1$ the leading terms become $1 + k\rho^2/2b$ (for $\beta \ll 1$ we find $1 + \rho^2/2b^2$). Thus there is close correspondence, in the focal plane, between the Gaussian beam and the exact solution if $kb \gg 1$ and $\rho^2 \ll b/k$.

To sum up the correspondence between $\psi_G(\rho,z)$ and $\psi_b(\rho,z)$: it is close within the region (surrounding the focus) defined by $\left\{|z| \ll kb^2, \rho^2 \ll b/k\right\}$ provided $kb \gg 1$. Fig. 3 shows the ratio of the moduli $\left|\psi_G(\rho,z)/\psi_b(\rho,z)\right|$ in the focal region, for $kb = 2$.

4. Proof that an exact match to $\psi_G(0,z)$ does not exist

It is known ([2], Section 20.1.1) that any solution of the Helmholtz Eq. (1) which is independent of the azimuthal angle may be written as

$$\psi(\rho,z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, g(z + i\rho\cos\theta)\, e^{-ik\rho\sin\theta}. \tag{22}$$

On the beam axis $\psi(0,z) = g(z)$. Restricting ourselves to causally propagating beams (no backward propagation far from the focal region [11]) gives $\psi(\rho,z)$ of the form (6) with $m = 0$. Equating the axial value to the Gaussian beam value, we have

$$\psi(0,z) = g(z) = \int_0^k dq f(k,q)\, e^{q(b+iz)} = \frac{b}{b+iz}e^{ikz}. \tag{23}$$

The last equality is an integral equation for the function $f(k,q)$ (which could in addition depend on parameters such as the length b). Without loss of generality we may write $f(k,q) = be^{-kb}h(k,q)$. Then

$$\int_0^k dq\, h(k,q)\, e^{q(b+iz)} = \frac{e^{k(b+iz)}}{b+iz}. \tag{24}$$

In terms of the dimensionless variables $\xi = k(b+iz), \eta = q(b+iz)$ this Volterra integral equation of the first kind becomes

$$e^{\xi} = \int_0^{\xi} d\eta\, h\left(\frac{\xi}{b+iz}, \frac{\eta}{b+iz}\right) e^{\eta}. \tag{25}$$

A solution independent of z is required, so $h(k,q)$ has to be a function of one variable, the ratio q/k. Thus (25) reads

$$e^{\xi} = \int_0^{\xi} d\eta\, h\left(\frac{\eta}{\xi}\right) e^{\eta}, \text{or, with} \eta = \xi t, \frac{e^{\xi}}{\xi} = \int_0^1 dt\, h(t)\, e^{\xi t}. \tag{26}$$

We shall argue that (26) has no physical solution. The function $h(t)$ would have to be such that $\int_0^1 dt h(t)$ diverges, while $\int_0^1 dt h(t) e^{\xi t}$ does not, for any $\xi \neq 0$. This makes a physical solution unlikely. To proceed further we note that ξ is a complex variable, and that the left-hand side of the last equation has a simple pole at the origin. The contour integral around the origin of the complex ξ plane gives

$$\oint d\xi \frac{e^{\xi}}{\xi} = 2\pi i. \tag{27}$$

As long as $h(t)$ is smooth enough (which is needed for a physical beam with finite invariants), we can change the order of integration in the double integral on the right-hand side:

$$\oint d\xi \int_0^1 dt\, h(t)\; e^{\xi t} = \int_0^1 dt\, h(t) \oint d\xi\, e^{\xi t} = \int_0^1 dt\, h(t)\; 0 = 0. \tag{28}$$

We have thus reached a contradiction, which implies that no solution of the required form exists.

The same conclusion follows by considering the Laplace transform of Eq. (26) at $z = 0$, written in the form $e^{\beta} = \beta \int_0^1 dt h(t) e^{\beta t}, \beta = kb$. Taking Laplace transforms with respect to β of both sides, we obtain, for $s > 1$,

$$\int_0^{\infty} d\beta\, e^{\beta(1-s)} = \int_0^1 dt\, f(t) \int_0^{\infty} d\beta\, \beta\, e^{\beta(t-s)} \quad \text{or} \\ \frac{1}{s-1} = \int_0^1 dt \frac{f(t)}{(s-t)^2}. \tag{29}$$

The leading asymptotic terms for large s are s^{-1} on the left-hand side, and s^{-2} on the right-hand side. Again we have reached a contradiction.

5. An exact solution with Gaussian profile in the focal plane does not exist

The general form of a causally propagating cylindrically symmetric beam is (on changing the variable of integration in (5) from q to $\kappa = \sqrt{k^2 - q^2}$)

$$\psi(\rho, z) = \int_0^k d\kappa\, \kappa\, h(k,\kappa)\; e^{iz\sqrt{k^2-\kappa^2}} J_0(\kappa\rho). \tag{30}$$

The extra factor of κ has been inserted for convenience in the application of Hankel transforms. In the focal plane $z = 0$ this is to be of Gaussian form:

$$\psi(\rho, 0) = \int_0^k d\kappa\, \kappa\, h(k,\kappa)\; J_0(\kappa\rho) = \exp\left(-\frac{k\rho^2}{2b}\right). \tag{31}$$

Taking the Hankel transform of (31) by operating with $\int_0^{\infty} d\rho\rho J_0(\kappa'\rho)$, we get

$$\int_0^k d\kappa\, \kappa\, h(k,\kappa) \int_0^{\infty} d\rho\, \rho\, J_0(\kappa'\rho)\, J_0(\kappa\rho) \\ = \int_0^{\infty} d\rho\, \rho\, J_0(\kappa'\rho) \exp\left(-\frac{k\rho^2}{2b}\right). \tag{32}$$

The Hankel transform of a Gaussian is proportional to a Gaussian (formula 10.22.51 of [13]),

$$\int_0^{\infty} d\rho\, \rho\, J_0(\kappa\rho) \exp\left(-\frac{k\rho^2}{2b}\right) = \frac{b}{k} \exp\left(-\frac{b\kappa^2}{2k}\right). \tag{33}$$

The Hankel transform of a Bessel function is proportional to a Dirac delta,

$$\int_0^{\infty} d\rho\, \rho\, J_0(\kappa'\rho)\, J_0(\kappa\rho) = \kappa^{-1}\delta(\kappa' - \kappa). \tag{34}$$

Let $U(k,\kappa)$ be unity for $0 \le \kappa \le k$, and zero otherwise. Thus (32) reads, on dropping the prime on κ',

$$U(k,\kappa)\, h(k,\kappa) = \frac{b}{k} \exp\left(-\frac{b\kappa^2}{2k}\right). \tag{35}$$

The left-hand side is nonzero only in the interval $0 \le \kappa \le k$, while the right-hand side is nonzero on the entire positive real line. Thus there is no solution. To have a causal and non-divergent solution of the Helmholtz equation, we restrict the range in (30) to the interval $(0, k)$, but the resultant Hankel transform is not that of a Gaussian.

6. Discussion

The 'paraxial' Gaussian beam waveform (2), and its generalizations the Laguerre–Gauss and Hermite–Gauss waveforms, have been widely used. However, they are known to fail for strongly focused beams. It is interesting that, despite (2) being the solution of a paraxial equation, and thus presumed to be accurate close to the beam axis, the fact is that no exact solution of the Helmholtz equation can reproduce the Gaussian beam waveform on the axis. Thus the 'paraxial' Gaussian beam cannot give an exact representation of physical beams, even on the beam axis. It does remain a useful approximation within the focal region provided $b^2 + z^2 \gg \rho^2$ and $k^2(b^2 + z^2) \gg 1$, as we saw in Section 1. (These inequalities are satisfied when $kb \gg 1, \rho \ll \sqrt{b/k}, |z| \ll kb^2$, in accord with the results of Section 3.)

In Appendix B we show that the simplest Bessel–Gauss beam also cannot be the basis of an exact solution. This is more surprising, since the Bessel–Gauss beam does have the expected zeros in the focal plane.

In Appendix C we discuss the transformation of solutions of the paraxial equation into solutions of the Helmholtz equation, in parallel to a method proposed by Wünsche [6]. We show that the resultant 'solutions' have non-physical divergences.

As regards the exact solution $\psi_b(\rho, z)$, we note that, by construction, it contains no 'evanescent' waves (namely those decaying or growing exponentially with z). This is because the variable q which appears as e^{iqz} in the integrand defining $\psi_b(\rho, z)$ is real in its entire range. The absence of evanescent waves is in contrast to much literature on beams in which $q = k_z = \sqrt{k^2 - k_x^2 - k_y^2}$ is allowed to be imaginary, leading to exponential growth or decay. Exponential growth with z is not physical. It can be avoided by choosing different branches of the square root for positive and negative z, or equivalently by replacing e^{iqz} by $e^{iq|z|}$. However, this leads to backward propagation for negative z, and the resulting wavefunction is no longer a solution of the Helmholtz equation at $z = 0$, because the derivative of $|z|$ is a step function, and the second derivative a delta function. Evanescent waves exist, for example, within the medium of smaller refractive index in total reflection, but have no part in free space propagation far from material boundaries, in our view.

Appendix A. Divergence angle of the $\psi_b(\rho, z)$ beam

We wish to examine how the beam expands away from the axis. The behaviour is complicated within the focal region, but simplifies when $|z| \gg b, r \gg b$ and $kr \gg 1$. Let us find the asymptotic form of the unnormalized wavefunction, which from (11) is

$$\Psi(\rho, z) = 2iP_1\left(\frac{z - ib}{R}\right) j_1(kR) + \sum_0^{\infty} A_n P_{2n}\left(\frac{z - ib}{R}\right) j_{2n}(kR). \tag{A1}$$

Since $R^2 = r^2 - 2ibz + b^2$, we have $R \approx r - ibz/r$. The argument of the Legendre polynomials is therefore $z/r = \cos\theta$ plus correction terms of order b/z and bz/r^2, which we can neglect. However, $kR \approx kr - ikbz/r$, and the imaginary part is important because it leads to hyperbolic terms in the spherical Bessel functions, for which the asymptotic forms for large $|\zeta|$ are

$$j_1(\zeta) \to -\frac{\cos\zeta}{\zeta}, j_{2n}(\zeta) \to (-)^n \frac{\sin\zeta}{\zeta}. \tag{A2}$$

We also know from equation (3.20) of [11] that

$$\sum_0^{\infty} (-)^n A_n P_{2n}(\cos\theta) = 2|\cos\theta|. \tag{A3}$$

Hence the asymptotic form of (A1) is, on setting $kR \approx kr - ikb\cos\theta$,

$$\frac{2}{kr}\left\{-i\cos\theta\cos(kr - ikb\cos\theta) + |\cos\theta|\sin(kr - ikb\cos\theta)\right\}$$
$$= -\frac{2i\,|\cos\theta|}{kr}\exp\left[ikr\,\mathrm{sgn}(\cos\theta) + kb\,|\cos\theta|\right]. \tag{A4}$$

The asymptotic ratio of the moduli (off-axis to on-axis) of Ψ is therefore

$$|\Psi(\rho,z)/\Psi(0,z)| \to \cos^2\theta\, e^{-kb(1-|\cos\theta|)}. \tag{A5}$$

When kb is large compared to unity the exponent in (A5) is -1 when $\theta \approx \sqrt{2/kb}$, which is the same as for the Gaussian beam for large kb, given below Eq. (4). For comparison, for ψ_G the asymptotic ratio of the moduli (off-axis to on-axis) is $e^{-(kb/2)\tan^2\theta}$, quite different from (A5) except at small θ.

Appendix B. An exact solution reproducing the simplest Bessel–Gauss beam in the focal plane does not exist

Bessel–Gauss beams, in their simplest azimuthally symmetric form, are defined by the focal plane wavefunction being set equal to that of the Gaussian beam times a Bessel function of order zero [14]:

$$\psi(\rho,0) = e^{-k\rho^2/2b} J_0(k\rho). \tag{B1}$$

Comparison with exact solutions of the form (30) gives the equality, on $z = 0$,

$$\psi(\rho,0) = \int_0^k d\kappa\,\kappa\,h(k,\kappa)\,J_0(\kappa\rho) = e^{-k\rho^2/2b} J_0(k\rho). \tag{B2}$$

The function $h(k,\kappa)$ is to be found, if possible. Let $U(k,\kappa)$ be unity for $0 < \kappa < k$, and zero otherwise. Then (B2) is the Hankel transform of $U(k,\kappa)\,h(k,\kappa)$:

$$\psi(\rho,0) = \int_0^\infty d\kappa\,\kappa\,U(k,\kappa)\,h(k,\kappa)\,J_0(\kappa\rho). \tag{B3}$$

Taking the inverse Hankel transform of (B2) by operating with $\int_0^\infty d\rho\rho J_0(\kappa'\rho)$, and using (34), we get (dropping the primes on κ)

$$U(k,\kappa)\,h(k,\kappa) = \int_0^\infty d\rho\,\rho\,e^{-k\rho^2/2b} J_0(\kappa\rho)\,J_0(k\rho). \tag{B4}$$

The right-hand side of (B4) is a special case of Weber's second exponential integral ([15], Section 13.31), and is equal to

$$\frac{b}{k}\exp\left(\frac{b\left(k^2+\kappa^2\right)}{2k}\right) I_0(\kappa b). \tag{B5}$$

Thus we again have a contradiction, since $U(k,\kappa)\,h(k,\kappa)$ is nonzero only when $0 < \kappa < k$, while the expression (B5) is not.

Appendix C. Converting solutions of the paraxial equation into solutions of the Helmholtz equation

Wünsche [6] has found operators which formally convert solutions of the paraxial equation into solutions of the Helmholtz equation. We shall give an equivalent method here, and at the same time demonstrate the weakness of such constructions.

The (approximate) paraxial equation is, from Section 1 and for axially-symmetric solutions,

$$\left(\partial_\rho^2 + \rho^{-1}\partial_\rho + 2ik\partial_z\right) G \approx 0, \qquad \psi(\rho,z) = e^{ikz} G(\rho,z). \tag{C1}$$

The paraxial equation is solved by $G = e^{-iqz} J_0\left(\rho\sqrt{2kq}\right)$ for any q. Superposition with weight function $k^{-1}g(k,q)$ gives the general (regular) paraxial solution,

$$G(\rho,z) = k^{-1}\int_0^\infty dq\,g(k,q)\,e^{-iqz} J_0\left(\rho\sqrt{2kq}\right). \tag{C2}$$

The substitution $q = \kappa^2/2k$ gives us

$$G(\rho,z) = k^{-2}\int_0^\infty d\kappa\,\kappa\,g\left(k,\kappa^2/2k\right) e^{-iz\kappa^2/2k} J_0(\kappa\rho). \tag{C3}$$

As an example, let us match (C3) to the Gaussian beam (2), for which

$$G(\rho,z) = \frac{b}{b+iz}\exp\left\{-\frac{k\rho^2}{2(b+iz)}\right\}. \tag{C4}$$

To find the corresponding function g, we take the Hankel transform of (C4) equated to (C3), using (33) and (34): operation with $\int_0^\infty d\rho\rho J_0(\kappa'\rho)$ gives

$$\frac{b}{k} e^{-\frac{\kappa'^2}{2k}(b+iz)} = k^{-2} g\left(k,\kappa'^2/2k\right) e^{-iz\kappa'^2/2k}. \tag{C5}$$

Hence the weight function corresponding to the fundamental Gaussian mode is $g\left(k,\kappa^2/2k\right) = kbe^{-b\kappa^2/2k}$; substitution in (C3) gives $G = \frac{b}{b+iz}\exp\left\{-\frac{k\rho^2}{2(b+iz)}\right\}$. Note there is no cut-off in the integration at $\kappa = k$. As in Section 5, such a weight function leads to an exact solution of the Helmholtz equation of the form

$$\psi(\rho,z) = \frac{b}{k}\int_0^\infty d\kappa\,\kappa\,e^{-b\kappa^2/2k} e^{iz\sqrt{k^2-\kappa^2}} J_0(\kappa\rho). \tag{C6}$$

This integral reproduces the Gaussian beam waveform in the focal plane. On the beam axis the integral can be evaluated in terms of error functions. We find that it grows without bound on the beam axis for negative z large compared to $\sqrt{2b/k}$, in proportion to $ze^{kz^2/2b}$. Thus (C6) is not a physical wavefunction. On the other hand, if the integral in (C6) is terminated at $\kappa = k$, the resulting wavefunction is a well-behaved member of the generalized Bessel beam family [11], and in fact reduces to the proto-beam of [11] when $kb \to 0$.

The same problem consequent (in our approach) from infinite range in the integral, arises in Wünsche's method of converting solutions of the paraxial equation into solutions of the Helmholtz equation, but in another guise. In evaluating the correction to the Gaussian beam on the beam axis, Wünsche ([6], Section 5 and Appendix B) also finds the error functions which result from setting $\rho = 0$ in (C6).

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