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Comparing Income Distributions Using Atkinson's Measure of Inequality.*

John Creedy†

Abstract

This paper is aimed at undergraduate and graduate economics students, and public sector economists, who are interested in inequality measurement. It examines the use of the Atkinson inequality measure to compare income distributions. A major feature of this measure is that distributional value judgements are made explicit, via the use of a particular form of Social Welfare Function. Emphasis is given to the interpretation of changes in inequality and the role of the relative inequality aversion parameter, which reflects an important feature of those value judgements.

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Keywords: Inequality; Atkinson Measure; Inequality Aversion; Excess Share.

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1 Introduction

The increased interest in inequality in recent years has led to considerable attention being given to changes in summary measures of inequality. The most widely used measure in offical statistics and public debates is the Gini index, devised in 1914 by the Italian statistician Corrado Gini (1884-1965).[1] The popularity of Gini's index may perhaps be related to the ease with which it can be described in terms of the famous Lorenz curve diagram, which plots the cumulative proportion of total income against the corresponding proportion of people, after ranking all incomes in ascending order: it is simply twice the area between the Lorenz curve and the diagonal line of equality.

Despite the early suggestion, by Dalton (1920), that inequality measurement should involve an explicit statement of the values involved in making judgements about income differences, it was not until 56 years after Gini's paper that this was achieved in a major contribution by the British economist Tony Atkinson (1944-2017). His 1970 paper transformed the way economists think about inequality measurement, and stimulated a vast literature which changed the subject from the statistical analysis of a measure of dispersion to a central topic in welfare economics. It is undoubtedly his best-known paper, and was an early signpost to a remarkable career. As remarked by Brandolini *et al.* (2018, p. 181) in their affectionate memoir, he 'was one of the world's leading economists, an unrivalled scholar of inequality and poverty and among the founders of the modern study of public economics'.

The great value of Atkinson's (1970) analysis is that he based his measure on explicit value judgements, assumed to be held by a hypothetical independent judge, and summarised by a social welfare, or evaluation, function by which an income distribution is assigned a cardinal value. As explained more fully below, Atkinson measured inequality in terms of the proportional difference between two income values. These are the arithmetic mean income, and the income level, called the 'equally distributed equivalent' income, which, if obtained by everyone, produces the same value of 'social welfare' as the actual distribution. Stimulated by contemporary literature on risk aversion and increasing risk, he used a particular form of welfare function that involves a single parameter, reflecting the 'relative inequality aversion' of the judge, similar to relative risk aversion in the uncertaintly context. Atkinson's paper stimulated a search

[1] For a review of Gini's contributions to inequality measurement, see Forcina and Georgi (2005).

for the value judgements that are implicit in other measures, including the Gini index: it turns out that the Gini measure can also be expressed in terms of the ratio of arithmetic mean income to an equally distributed equivalent value, but with a social welfare function expressed as an inverse-rank-weighted sum of incomes.[2] The welfare function used by Atkinson, as seen below, depends only on relative incomes, not their ranks in the distribution or on absolute income differences.

It is unfortunate that, over 50 years since Atkinson's contribution, his measure is largely confined to specialist economic literature. Gini's measure remains ubiquitous in public debates, with virtually no mention of the implicit value judgements involved. The important lesson – that it is necessary to investigate the implications of using different value judgements – is ignored. One explanation for this state of affairs is that there is undoubtedly a communication challenge faced by those who report inequality measures to a wider audience, and it is likely that many of those who report Gini measures in popular debates are themselves ignorant of the implicit values involved. In the case of Atkinson's measure, there is certainly a challenge in appreciating and communicating to a wider audience the meaning that can be attached to a change in the index. For example, is a five per cent increase in an inequality measure a major concern for an inequality-averse judge, or may it be regarded as quite small? It is relatively easy to form a view about a change in a simple aggregate income measure, but it is intuitively much more difficult to appreciate orders of magnitude of a summary measure of many individual values.[3] Atkinson, in his original contribution, provided some guidance regarding the degree of inequality aversion used in calculating his measure, by considering a 'leaky bucket' thought experiment involving the 'leak' that a judge is prepared to tolerate in making a transfer from a richer to a poorer person. Nevertheless, even among professional economists, measures using completely inappropriate values continue to be reported.

The present paper is therefore aimed at students, public servants, and others who wish to make inquality comparisons among population groups or over time and, im-

[2]Hence, the largest income is given a weight of 1, while the lowest income has a weight equal to the population (strictly, the sample) size.

[3]If concern is only with the question of whether inequality has increased or decreased, then of course a first indication is provided by examining Lorenz curves. If these do not intersect, a comparison can unambiguously be made involving only the basic value judgement encapsulated in the 'principle of trasfers', discussed below. However, intersecting curves are more common in practice.

portantly, face a need to communicate results to non-specialists. This attention to some less well-known features of a single measure is warranted, given the special characteristics of Atkinson's measure and the complexities involved, although some of the approaches can be applied to other inequality measures. There is no 'correct' way to measure inequality, as Atkinson stressed, but it is important for potential users of a measure to have an appreciation of its main features. The Atkinson measure can be used to provide a range of values, depending on assumed inequality aversion, but he would be the first to stress that it represents only one particular type of value judgement. This paper explains some possible methods of providing explanations of what a given value of the index, or a change in it, implies in terms that can be more easily understood. Furthermore, the general implications of varying the imposed degree of inequality aversion are discussed.

First, the Atkinson measure is defined in Section 2. It is seen that it takes the perspective of an independent judge with a specified form of value judgements regarding income comparisons among individuals. It implies a well-defined 'marginal rate of substitution' between equality and total income (or, as it is sometimes expressed, between 'equity and efficiency'). This makes it possible to talk about the extent of total income growth that would be foregone by a judge in order to achieve a given reduction in inequality.

Another possible approach to interpretation, discussed in Section 3, is to consider an alternative artificial distribution that has the same value of the inequality measure as the actual distribution but consists of only two income levels (or, equivalently, income shares), though there may be more than two individuals in the artificial distribution.[4] A basic analogy with a 'cake-cutting exercise' is therefore involved.

The simple observation that a small income increase for a high-income person leads to an increase in inequality, but an increase for a low-income person results in a reduction in inequality, gives rise to the idea of a particular income level that in some sense divides the low and high incomes. The concept is examined in Section 4 of a 'pivotal' individual, such that if an additional small income increase is given to a poorer individual, the Atkinson inequality index falls.[5]

[4]This approach was proposed by Subramanian (2002), and extended by Shorrocks (2005), in the constext of the Gini measure.

[5]Corvalan (2014) proposed this concept in the context of the Gini measure. Related ideas include Hoffmann's (2001) use of a dividing line, defined as the 'relative poverty line', and Lambert and

Atkinson showed that if Lorenz curves do not intersect, his measure gives the same ordering of distributions for all values of the degree of inequality aversion, using the class of welfare functions involved.[6] However, such a clear cut result rarely arises when comparing distributions, and he showed how the ranking of a number of countries varies substantially as the degree of inequality aversion is increased. The question arises of whether anything more specific can be said about whether two distributions are likely to result in intersecting profiles of the Atkinson inequality measure as aversion increases. It is shown in Section 5 that the relationship can be described in terms of several elasticities and other characteristics of the income distributions concerned.

A basic property shared by all single summary measures of a distribution is that there can be a wide range of changes within the distribution that, overall, are consistent with an unchanged summary measure. Such inequality-preserving changes are discussed in Section 6. Conclusions are in Section 7.

In order to focus on these basic aspects, the discussion is in terms of single individuals, with 'income' measured over a given time period. This avoids important complications – common to the use of any inequality measure – associated with the choice of the appropriate unit of analysis (when households or families are examined), the accounting period, and the precise 'welfare metric' to use.[7] References to the vast literature are kept to a minimum here. However, broader discussions of Atkinson's seminal paper, and the subsequent vast literature, can be found in, for example, Lambert (1993) and Jenkins (2016).[8]

2 The Atkinson Inequality Measure

Consider n individuals with incomes of $y_1, y_2, ..., y_n$. In general, the distributional value judgements of an independent judge can be summarised by a social welfare function, expressed as $W = W(y_1, y_2, ..., y_n)$.[9] Aigner and Heins (1967) had used a welfare function approach to define an index of *equality*, defined as the social welfare resulting from

Lanza's (2006) 'benchmark income or position'.

[6]For further discussion of this important 'dominance' result, see Lambert (1993).

[7]The range of choices, and value judgements involved, are discussed by Creedy (2017b).

[8]Further technical details of analyses summarised here can be found in Creedy (2016, 2017a, 2019).

[9]The widely-used term 'social welfare' is somewhat misleading here, because W represents the judge's evaluation function, expressed in terms of the distribution. It does not refer to the welfare of individuals, or any aggregate based on individuals' utility.

the actual distribution, divided by the welfare from a distribution in which everyone receives the arithmetic mean income. They investigated a number of functional forms for W. However, given any form of this function, Atkinson pointed out that it is possible to define an equally-distributed-equivalent income level, y_E, such that:

$$W\left(y_E, y_E, ..., y_E\right) = W\left(y_1, y_2, ..., y_n\right) \tag{1}$$

That is, y_E is the income which, if obtained by each person, is considered by the judge as generating the same value of W as the actual distribution. A class of inequality measures, I_W can then be defined in terms of the proportional difference between arithmetic mean income, $\overline{y}$, and y_E, whereby:

$$I_W = 1 - \frac{y_E}{\overline{y}} \tag{2}$$

Clearly, a wide range of measures exists, depending on the form of W.[10] Atkinson chose to examine the implications of the function, $W = W_\varepsilon$, which depends on a single parameter, ε, and is additive, and concave, such that, for $\varepsilon \neq 1$:

$$W_\varepsilon = \left(\frac{1}{n}\right) \sum_{i=1}^{n} \frac{y_i^{1-\varepsilon}}{1-\varepsilon} \tag{3}$$

If $\varepsilon = 1$, it takes the form, $W_1 = \left(\frac{1}{n}\right) \sum_{i=1}^{n} \log\left(y_i\right)$. Here, ε reflects the degree of relative inequality aversion of the judge. This welfare function is individualistic, additive, and Paretean. It also satisfies the 'Principle of Transfers', often considered to be a basic value judgement shared by many people: this states that an income transfer from a richer to a poorer person, which does not affect their relative positions, represents an 'improvement', that is, an increase in W_ε.[11] Hence, in this case, $y_E = y_\varepsilon$ and for the Atkinson measure, $I_W = A_\varepsilon$, where y_ε is the power mean:

$$y_\varepsilon = \left\{\frac{1}{n} \sum_{i=1}^{n} y_i^{1-\varepsilon}\right\}^{1/(1-\varepsilon)} \tag{4}$$

[10] It was mentioned in the introduction that the Gini measure arises from a welfare function that is a weighted sum of individual incomes, using 'inverse rank' weights.

[11] However, this view is not necessarily shared by all judges. For example, if the transfer is from the richest person to the next-richest person, the resulting greater distance between the top two individuals and the rest may not be considered as equalising. Similarly, the Paretean value judgement may not be shared by all judges, as some may object to a gain which is experienced only by the richest person.

giving Atkinson's inequality measure as:

$$A_\varepsilon = 1 - \frac{y_\varepsilon}{\bar{y}} \tag{5}$$

Hence inequality is expressed in terms of a ratio of two measures of location (or central tendency), here the ratio of a power mean of order, $1-\varepsilon$, to the arithmetic mean, $\bar{y}$. In defining the measure here, no distinction was drawn between sample and population values.[12] However, the usual situation is that researchers have access to sample surveys, so it is useful to be able to compute standard errors: the sampling properties are discussed in the Appendix below.

In considering the values of ε to examine in empirical applications, Atkinson proposed a thought experiment involving a 'leaky bucket', and the question of the leak that would be tolerated by the judge in making a transfer (via the bucket) from a richer to a poorer person. Thus, totally differentiating W_ε with respect to two incomes, y_j and y_k, gives:

$$dW_\varepsilon = \frac{\partial W_\varepsilon}{\partial y_j} dy_j + \frac{\partial W_\varepsilon}{\partial y_k} dy_k \tag{6}$$

and for $dW_\varepsilon = 0$:[13]

$$\left.\frac{dy_j}{dy_k}\right|_{W_\varepsilon} = -\frac{\partial W_\varepsilon / \partial y_k}{\partial W_\varepsilon / \partial y_j} = -\left(\frac{y_k}{y_j}\right)^{-\varepsilon} \tag{7}$$

Hence if $y_k > y_j$, a given transfer from person k requires person j to receive a smaller amount, for the judge to be indifferent to the two distributions. This is because of the adherence to the Principle of Transfers, reflected in the concavity of the W_ε function, according to which, $\frac{\partial W_\varepsilon}{\partial y_k} < \frac{\partial W_\varepsilon}{\partial y_j}$. For example, if $y_k = 2y_j$ and $\varepsilon = 1$, and (in discrete terms) one dollar is taken from k (so that $\Delta y_k = -1$, it is necessary, for constant W_ε, to give only $\Delta y_j = 2^{-1} = 0.50$, or 50 cents, to person j. If $\varepsilon = 0.2$, a leak of 0.13 is tolerated.[14]

[12] When using sample data which contain a range of other characteristics of indivduals, 'calibration weights' are sometimes attached to each individual, reflecting differential responses. These ensure that certain totals reflect population values obtained from other sources, such as a census. The formula for Atkinson's measure is easily adapted to deal with such weights, in the same way that it is modified when dealing with grouped frequency distributions of income.

[13] The following formula also indicates the homothetic nature of the welfare function, and the fact that only relative incomes matter when making comparisons. It is possible to construct Atkinson-type measures using welfare functions for which absolute differences between incomes matter.

[14] Using a range of questions based on the leaky bucket experiment, Amiel *et al.* (1999) investigated the aversion to inequality of students in three countries.

Before examining further features of A_ε, it is first useful to express the welfare function behind Atkinson's measure in a different way. It is a simple step to rearrange (5) as $y_\varepsilon = \bar{y}(1 - A_\varepsilon)$ and, recognising that y_ε, obtained by everyone, gives rise to W_ε, an 'abbreviated welfare function', denoted W_ε^*, is:

$$W_\varepsilon^* = \bar{y}(1 - A_\varepsilon) \tag{8}$$

It is called an abbreviated function because, instead of being written as a function of all the incomes, it has just two arguments, $\bar{y}$ and $1 - A_\varepsilon$. It is therefore possible to think of indifference curves of the judge, which reflect the property of decreasing marginal rates of substitution, familiar from the utility analysis of individual's demands for two goods. Here the two goods are mean income and equality, $E_\varepsilon = 1 - A_\varepsilon$. The marginal rate of substitution is obtained by totally differentiating (8) and setting $dW_\varepsilon^* = 0$. Hence, dropping the ε subscript on A, W and E here and everywhere below for convenience:

$$\left.\frac{d\bar{y}}{dE}\right|_{W^*} = -\frac{\bar{y}}{E} \tag{9}$$

Converting this expression to one involving proportional changes in $\bar{y}$ and E shows that a given small proportional increase in $\bar{y}$ is equivalent, in social welfare terms, to the same proportional reduction in E. Or, put differently, the judge is prepared to sacrifice a given reduction in equality for the same proportional increase in mean income: the judge's 'trade-off' between 'equity and efficiency' is thus particularly simple.[15] Converting to terms involving A rather than E, it can be seen, for example, that an increase in A from 0.25 to 0.30, which represents a 20 per cent increase in inequality, would have to be compensated by an increase in arithmetic mean income of 6.67 per cent, for the judge to be indifferent to the change.

3 An Equivalent Small Distribution

It is difficult to envisage what a particular value of A implies in view of the fact that incomes are distributed among a large number of people. One approach to interpreting orders of magnitude is to consider an artificial small population having the same

[15] This is of course not unique to the Atkinson measure, as it has been mentioned that the Gini measure can also be expressed in terms of the ratio of mean income to an equally distributed equivalent income.

inequality measure. The approach asks what a distribution with just two income levels would look like, having the same A value. Hence, suppose there are just two types of individual and, for convenience only, the total income is normalised to one unit. As relative inequality is of concern, the income values can also be considered as income shares. One person has an income of y, and the remaining $n-1$ people have an income of $(1-y)\,/\,(n-1)$ each, and arithmetic mean income is $1/n$. Thus:

$$A = 1 - n\left[\frac{1}{n}\left\{y^{1-\varepsilon} + (n-1)\left(\frac{1-y}{n-1}\right)^{1-\varepsilon}\right\}\right]^{1/(1-\varepsilon)} \tag{10}$$

Given a value of A obtained from an actual income distribution, the aim is to solve (10) for y. Rearranging this, it can be seen that y is given by the root (or roots) of:

$$y^{1-\varepsilon} + (n-1)^{\varepsilon}(1-y)^{1-\varepsilon} - n\left(\frac{1-A}{n}\right)^{1-\varepsilon} = 0 \tag{11}$$

It may initially seem natural to consider just two individuals in the artificial population, with only a richer and a poorer person and thereby making the properties more transparent. However, if $n = 2$ it is not always possible to solve (11), remembering also that feasible solutions require $0 < y < 1$. Allowing for $n > 2$ can enable a feasible solution to be obtained.

Consider how the values of y vary as the Atkinson measure increases. Table 1 illustrates the effect of increasing A by 20 per cent, from an initial value of $A = 0.25$. A dash (–) in the table indicates that for the combination of n and ε, no feasible solution for y exists: it is not possible to generate the required inequality level with such a small number of individuals. The table shows that for the large (20 per cent) increase in A, the associated increase in the income of the rich person in the small-sample construction varies depending on the sample size, n, and the degree of inequality aversion. For $\varepsilon = 0.5$ and $n = 2$, the value of y increases by 3.2 per cent as A increases by 20 per cent. This increases to 6.7 per cent the same ε but for $n = 4$. For the low inequality aversion parameter of $\varepsilon = 0.2$ in combination with $n = 6$ and $n = 8$, y must increase by 6.7 and 7.0 per cent respectively. Yet for $\varepsilon = 0.9$ in combination with $n = 6$ and $n = 8$, y must increase by 7.2 and 17.4 per cent respectively.

An associated concept is that of the 'excess share' obtained by the rich person, defined as the difference between y and arithmetic mean income, $1/n$. Table 2 converts

Table 1: Top Income for Alternative Inequality Measures and Population Size

	$A = 0.25$				$A = 0.30$			
ε	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 2$	$n = 4$	$n = 6$	$n = 8$
0.2	–	–	0.90	0.85	–	–	0.96	0.91
0.5	0.93	0.75	0.66	0.60	0.96	0.80	0.71	0.65
0.9	0.84	0.63	0.55	0.46	0.87	0.68	0.59	0.54

the values of y in Table 1 into their corresponding excess shares, $y - 1/n$. One feature of these values is that, for given ε, the variation in the excess share as n increases is not monotonic. It increases initially as n increases from $n = 2$, and then declines slightly after $n = 4$. However, the excess share decreases monotonically as the degree of inequality aversion, ε, increases, for which more weight is attached to the lower income range. For a given value of n, the richer share, y, must decrease as ε increases, because it is easier to achieve the given inequality measure, A, with a lower share when aversion to inequality is higher. The excess share must also fall as ε increases, given that such comparisons hold n constant. The changes in the excess share vary as A increases from 0.25 to 0.30: it varies from about 8 per cent to over 20 per cent (for the $\varepsilon = 0.9$ combined with $n = 8$ case).

Table 2: Excess Shares for Alternative Inequality Measures and Population Size

	$A = 0.25$				$A = 0.30$			
ε	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 2$	$n = 4$	$n = 6$	$n = 8$
0.2	–	–	0.73	0.73	–	–	0.79	0.79
0.5	0.43	0.50	0.49	0.48	0.46	0.55	0.54	0.53
0.9	0.34	0.38	0.38	0.34	0.37	0.43	0.42	0.42

4 The Pivotal Income

A further concept used to add further interpretation to an inequality measure is the concept of the pivotal income, y^*, defined as the income below which a small increase leads to a reduction in inequality. For the Atkinson measure, it is given by the solution to $\partial A/\partial y_i = 0$.[16] Thus:

$$\frac{\partial A}{\partial y_i} = -\frac{y_\varepsilon y_i^{-\varepsilon}}{n\bar{y}} \frac{1}{\frac{1}{n}\left(\sum_{i=1}^{n} y_i^{1-\varepsilon}\right)} + \frac{y_\varepsilon}{n\bar{y}^2} \tag{12}$$

Setting this equal to zero and rearranging gives:

$$(y^*)^{-\varepsilon} = \frac{\frac{1}{n}\sum_{i=1}^{n} y_i^{1-\varepsilon}}{\bar{y}} \tag{13}$$

From (4) and (5):

$$\frac{1}{n}\sum_{i=1}^{n} y_i^{1-\varepsilon} = ((1-A)\,\bar{y})^{1-\varepsilon} \tag{14}$$

Thus the pivotal income is easily obtained as:

$$y^* = \bar{y}\,(1-A)^{\frac{\varepsilon-1}{\varepsilon}} \tag{15}$$

For example, if $\varepsilon = 0.5$ and $A = 0.3$, $y^* = 1.43\bar{y}$ and a small addition to the income of anyone with an income smaller than 1.43 times the arithmetic mean produces a reduction in A. Furthermore:

$$\frac{y^*}{y_\varepsilon} = (1-A)^{-\frac{1}{\varepsilon}} \tag{16}$$

so that in this case the pivotal income is 2.04 times the equally distributed equivalent income. Further examples of the sensitivity of the pivotal income are shown in Figure 1, which plots the ratio, $y^*/\bar{y}$, for variations in ε, given inequality measures of 0.6 and 0.2.

The variation in the ratio of the pivotal income to the arithmetic mean, as inequality increases for a given value of inequality aversion, is shown in Figure 2 for two different values of ε. It can be seen that the pivotal income is not very sensitive to variations in inequality for the higher value of inequality aversion, but is much more sensitive for lower ε.

5 Variations in Inequality Aversion

This section considers the question of whether anything specific can be said about whether two distributions are likely to result in intersecting profiles of the Atkinson

[16] The following result is a special case of the more general results obtained for 'non-positional measures' by Lambert and Lanza (2006).

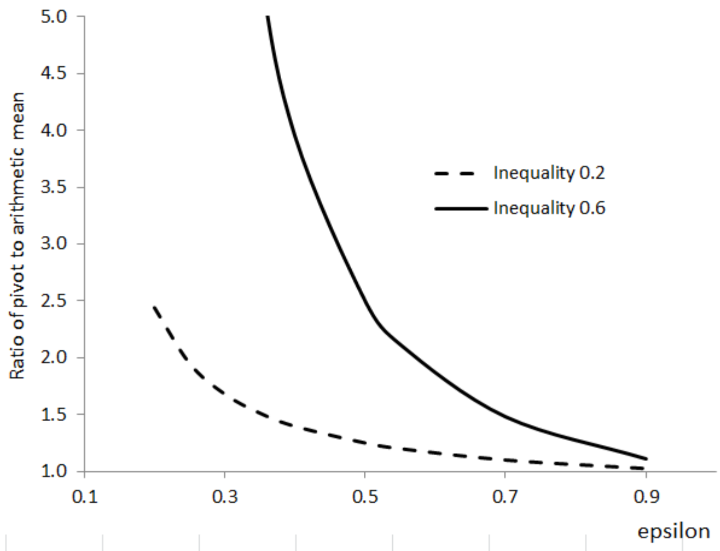

Figure 1: Ratio of Pivotal Income to Arithmetic Mean and Variation in Inequality Aversion

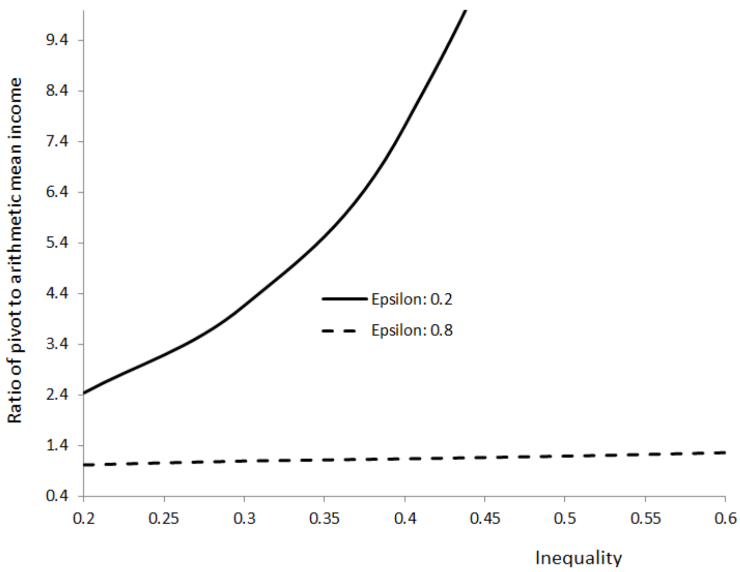

Figure 2: Variation in Pivotal Income with Inequality

inequality measure as aversion increases.[17] This is of particular concern when comparing the inequality effects of taxes and transfers. It is possible for a change in a tax and transfer system to be judged as inequality increasing or decreasing, depending on the degree of relative inequality aversion.[18] Deaton (1997, p. 155) suggests that, 'there is no simple relationship between the value of [inequality aversion] and the rankings [of two distributions]'. However, this section shows how conditions under which the ranking of two distributions changes, and the rate at which inequality of two distributions converges or diverges, can be expressed in terms of several new elasticities.

Direct differentiation of A with respect to ε is not straightforward. Consider, instead, looking at equality, $E = 1 - A = \frac{y_\varepsilon}{\bar{y}}$. Hence, $\log E = \log y_\varepsilon - \log \bar{y}$, and letting $a = 1 - \varepsilon$, differentiation with respect to a gives:

$$\frac{d \log E}{da} = \frac{d \log y_\varepsilon}{da} \tag{17}$$

The change in log-equality as a increases is therefore simply the change in the logarithm of equally distributed equivalent income. Multiplying both sides of (17) by a, and using the notation, $\eta_{z,y}$, to denote the elasticity of z with respect to y, gives (since $d \log y = dy/y$):

$$\eta_{E,a} = \eta_{y_\varepsilon,a} \tag{18}$$

Taking logarithm of y_ε from (4), so that $\log y_\varepsilon = \frac{1}{a} \log \left(\frac{1}{n} \sum_{i=1}^{n} y_i^a \right)$, this elasticity becomes :

$$\frac{d \log y_\varepsilon}{da} = -\frac{1}{a^2} \log \left(\frac{1}{n} \sum_{i=1}^{n} y_i^a \right) + \frac{1}{a} \frac{d \log \left(\frac{1}{n} \sum_{i=1}^{n} y_i^a \right)}{da} \tag{19}$$

This can be expressed more succinctly as:

$$\eta_{\log y_\varepsilon, a} = -1 + \eta_{\log \left(\frac{1}{n} \sum_{i=1}^{n} y_i^a \right), a} \tag{20}$$

To examine the elasticity in (20), consider $d \log \left(\frac{1}{n} \sum_{i=1}^{n} y_i^{1-\varepsilon} \right) / da$, and again using $d \log y = dy/y$:

$$\frac{d \log \left(\frac{1}{n} \sum_{i=1}^{n} y_i^a \right)}{da} = \frac{1}{\sum_{i=1}^{n} y_i^a} \frac{d \left(\sum_{i=1}^{n} y_i^a \right)}{da} \tag{21}$$

[17] In the special case where two distributions are lognormally distributed, the condition under which one distribution's Atkinson measure is greater than another depends only on the two variances of logarithms of income. However, while the lognormal often provides a good approximation, no actual distribution conforms precisely to a theoretical form.

[18] For an example of a policy reform leading to ambiguous results for some demographic groups, in the context of behavioural microsimulation, see Alinaghi *et al.* (2020).

Now consider the final term, $\frac{d\left(\sum_{i=1}^{n} y_i^a\right)}{da}$. In general, for constant, b, and variable, y, $\frac{d}{dy}\left(b^y\right) = b^y \log b$. Hence:

$$\frac{d\left(\sum_{i=1}^{n} y_i^a\right)}{da} = \sum_{i=1}^{n} y_i^a \log y_i \tag{22}$$

Substituting into (21) and writing in elasticity form gives:

$$\eta_{\log\left(\frac{1}{n}\sum_{i=1}^{n} y_i^a\right),a} = a \frac{\sum_{i=1}^{n}\left(y_i^a / \sum_{i=1}^{n} y^a\right)\log y_i}{\log\left(\frac{1}{n}\sum_{i=1}^{n} y_i^a\right)} \tag{23}$$

In general, elasticities of a variable and the logarithm of that variable are related by the simple relationship:

$$\eta_{y,z} = (\log y)\, \eta_{\log y,z} \tag{24}$$

Hence:

$$\begin{aligned} \eta_{y_\varepsilon,a} &= (\log y_\varepsilon)\, \eta_{\log y_\varepsilon,a} && (25)\\ &= (\log y_\varepsilon)\left\{-1 + a\frac{\sum_{i=1}^{n}\left(y_i^a / \sum_{i=1}^{n} y^a\right)\log y_i}{\log\left(\frac{1}{n}\sum_{i=1}^{n} y_i^a\right)}\right\} && (26)\end{aligned}$$

and using the fact that $y_\varepsilon = E\bar{y}$:

$$\eta_{E,a} = \eta_{y_\varepsilon,a} = \sum_{i=1}^{n}\left(\frac{y_i^a}{\sum_{i=1}^{n} y_i^a}\right)\log y_i - (\log \bar{y}E) \tag{27}$$

This elasticity has a convenient interpretation. The proportional change in equality, resulting from a proportional change in inequality aversion, is the difference between a weighted average of log-income and the logarithm of 'equality adjusted' arithmetic mean income (that is, the logarithm of equally distributed equivalent income).

Furthermore, (27) can be converted into an elasticity of A with respect to ε, as follows:

$$\eta_{A,\varepsilon} = \eta_{E,a}\left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{1-A}{A}\right) \tag{28}$$

Similarly, the elasticity of E with respect to ε is related to $\eta_{E,a}$ using:

$$\eta_{E,\varepsilon} = -\eta_{E,a}\left(\frac{\varepsilon}{1-\varepsilon}\right) \tag{29}$$

In the special case of $a = 0$, $\sum_{i=1}^{n}\left(\frac{y_i^a}{\sum_{i=1}^{n} y_i^a}\right)\log y_i = \frac{1}{n}\sum_{i=1}^{n}\log y_i$, which is the logarithm of geometric mean income. Similarly, E is the ratio of the geometric mean income to arithmetic mean, so that $\log \bar{y}E$ is also equal to the logarithm of geometric mean. Hence $\eta_{E,a=0} = 0$ when $\varepsilon = 1$. Alternatively, when $a = 1$, corresponding to $\varepsilon = 0$, substitution into (27) gives:

$$\eta_{E,a=1} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i}{\bar{y}}\right)\log y_i - (\log \bar{y}) \tag{30}$$

This result shows that $\eta_{E,a=1}$ is the difference between a share-weighted mean log-income and the logarithm of arithmetic mean income. This is positive, so that $\eta_{E,a}$ begins positive for low ε and becomes negative for $\varepsilon > 1$. This can also be seen by returning to the simple relationship between elasticities, whereby the term, $\eta_{E,a} = \eta_{E,1-\varepsilon}$ is also expressed in terms of $\eta_{E,\varepsilon}$ as:

$$\eta_{E,a} = -\left(\frac{1-\varepsilon}{\varepsilon}\right)\eta_{E,\varepsilon} \tag{31}$$

Clearly $\eta_{E,\varepsilon}$ is negative for $\varepsilon > 0$: inequality is necessarily higher as inequality aversion increases. Hence $\eta_{E,a} > 0$ when $\varepsilon < 0$, and $\eta_{E,a} < 0$ when $\varepsilon > 1$.

The typical shapes of the various profiles can be illustrated by taking a simple numerical example. Suppose there are just 8 individuals, with incomes in ascending order given by: 5, 10, 20, 50, 100, 300, 500, 1000. Figure 3 shows how A and E vary as ε is increased. Figure 4 plots the three elasticities, $\eta_{1-A,1-\varepsilon}$, $\eta_{A,\varepsilon}$ and $\eta_{1-A,\varepsilon}$, as ε varies. The more extensive distributions found in practice will nevertheless give rise to similarly shaped smooth profiles.

Having derived a number of elasticities and considered their shapes, the question is then whether they can be used to say anything specific about the properties of the distributions for which the inequality ranking changes as inequality aversion increases. In the case of the simple distribution used in the previous subsection, a change involving an increase in one lower income (say increasing 20 to 25, or raising 50 to 60), but not the lowest income, does reduce inequality for lower values of inequality aversion. But for higher degrees of aversion (3.9 and 2.25 respectively for the two examples) the reduced weight given to those lower income implies that inequality increases. An intersection of the profiles of A against ε, for lower values of ε, can also be achieved by changing two incomes in the bottom tail of the distribution. Thus reducing 5 to 4, and at the same time raising 10 to 15, reduces inequality for $\varepsilon < 1.55$, after which A is higher than in the first distribution. The analysis provides insights into the precise variation in A

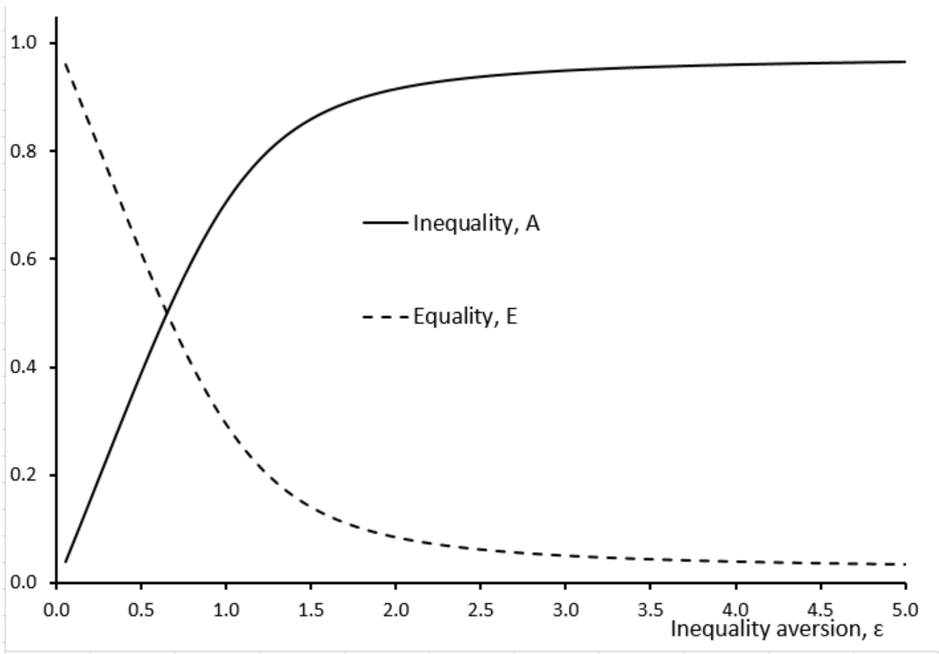

Figure 3: Variation in Inequality and Equality with Inequality Aversion

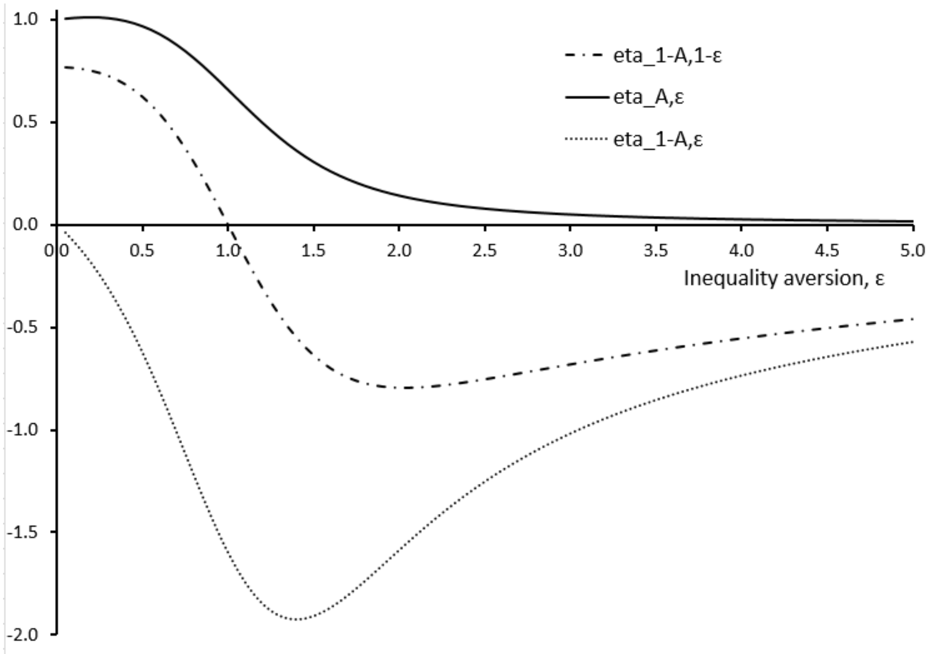

Figure 4: Variation in Elasticities With Inequality Aversion

with ε, expressed in terms of elasticities which have clear interpretations. Nevertheless, it is also seen that full information is needed about the income distribution if specific conditions for intersecting profiles of A against ε are to be determined. Hence, no easily-applicable rule can be established to determine if re-ranking will occur in a particular range of ε. It is therefore important to consider a sufficiently wide range of values of ε when making comparisons.

6 Inequality-Preserving Income Changes

A feature of any single summary measure of a large number of values is that the same numerical value of the measure can be consistent with a very broad range of distributions. This does not matter to a judge with the particular value judgements used in calculating A: it means that equalising changes in one part of the distrubution are just matched by disequalising changes in another part of the distribution. However, those who do not share the values behind reported stable inequality measures cannot be expected to believe that inequality has not changed over the relevant period.

The question arises of whether many inequality-preserving changes within an income distribution exist. Suppose it is required to distribute a fixed amount of money (income) among a given number of people, such that the resulting inequality measure takes a specified value.[19] Does this imply a unique income distribution? From one perspective, this is a trivial problem. Imposing one or more moments of a distribution, along with an inequality measure, simply specifies a set of linear and nonlinear constraints on the values in the distribution. So long as the number of individuals exceeds the number of constraints, there are some degrees of freedom in selecting values, and consequently the simultaneous equations do not necessarily have a unique solution.

Starting from some arbitrary distribution, an inequality measure can be preserved if an equalising transfer in one range of the distribution is appropriately matched by a disequalising transfer elsewhere. For simplicity, consider just three individuals. Suppose only the mean is imposed, so that the sum, $L = y_1 + y_2 + y_3$, is fixed, along with the equally distributed equivalent income, y_ε, implying a fixed value of A. Using $y_1 = L - y_2 - y_3$ and the definition of y_ε, the value of y_2, for given values of y_3, L and

[19] Since the Atkinson index is a relative measure, the actual fixed amount is not relevant.

y_ε, is given by the root or roots of:

$$(L - y_2 - y_3)^{1-\varepsilon} + y_2^{1-\varepsilon} + y_3^{1-\varepsilon} - 3y_\varepsilon^{1-\varepsilon} = 0 \tag{32}$$

This expression can have no real roots, one root or two real, positive and distinct roots. For example, suppose that $L = 6$ so that $\bar{y} = 2$. For $\varepsilon = 0.3$, $y_\varepsilon = 1.8$. Setting $y_3 = 3$, there are two solutions for y_2, equal to 0.2 and 2.8. However, the symmetry gives rise to corresponding values of y_1 of 2.8 and 0.2. Having specified the value of y_3, there is thus only one distribution, given by $[0.2, 2.8, 3.0]$, consistent with the imposed value of y_ε and the arithmetic mean. However, simply by setting y_3 to alternative values, a range of alternative distributions clearly exists for which the mean and Atkinson inequality measure are fixed. For example, starting from the above distribution, suppose that person 2 transfers 1.4 to person 3 and 0.4 to person 1. This combines an equalising with a disequalising transfer and results in the distribution $[0.6, 1.0, 4.4]$, which has the same Atkinson measure as the initial distribution.

With extra individuals, and hence an increase in the number of degrees of freedom, it is possible to generate an even wider range of possibilities. An equalising transfer in one range of the distribution can more easily be combined with a disequalising transfer in another range of the distribution, involving different pairs of individuals. Importantly, the existence of a wide range of inequality-preserving changes holds for all summary measures, not just the Atkinson measure. Hence, it is important that people viewing empirical results understand the nature of the value judgements behind the measure used: only if they agree with those judgements are they prepared to accept without question the view that a constant value implies no change in inequality.

7 Conclusions

In view of the strong interest in inequality and changes in inequality over time, and the widespread reference to inequality measures in popular debates, both in evaluating policy changes and in making a case for particular types of policy change, it is important to provide empirical results that give transparent guidance about orders of magnitude. This is especially challenging, particularly when different types of value judgement are involved, and inequality measures reduce values for many heterogeneous individuals to a single dimension.

This paper has considered this challenge in the context of the Atkinson inequality measure. One perspective is based on the abbreviated welfare function: given an annual growth rate, it could be said that a judge is prepared to give up a certain number of years of growth to obtain a given percentage reduction in inequality. Another point of view is obtained by converting the distribution to just a small number of values, having the same inequality as the actual distribution.The income of the richest person, measured in excess of the overall arithmetic mean, may provide another kind of simple illustration. However, this paper has shown that it is necessary to solve a nonlinear equation in order to calculate the income share values for any given value of Atkinson's index. Hence, it is not clear that the approach can provide the kind of information that non-specialists can easily digest in comparing different values of inequality, at least without some discussion, including an indication of the sensitivities involved. But with careful use they may provide useful supplementary descriptions. Such approaches do at least preserve the basic value judgements behind the inequality measures and allow for the implications of alternative value judgements to be investigated. This is preferred to the use of oversimplified comparisons, and measures used for rhetorical purposes, that actually disguise the implicit value judgements of those reporting results. An additional, and quite different, kind of insight is provided by the concept of the pivotal income. which is easily calculated given the Atkinson measure and associated degree of inequality aversion.

Although Atkinson stressed that inequality rankings are likely to vary as the degree of inequality aversion is varied, it turns out that there are no simple conditions relating to the distributions in question. A number of elasticities, with respect to inequality aversion, were derived and were shown to have convenient interpretations. Intersections of profiles of the Atkinson measure (plotted against the inequality aversion parameter) can arise without the need for pathological assumptions about the income distribution. Many distributional changes, involving higher inequality in the lower-income ranges of the distribution, are generally capable of producing intersections. Thus, it is necessary to consider a range of aversion parameters when examining an actual or proposed change to the tax and transfer system. By considering only one or two values, it could be concluded incorrectly that a tax reform is progressive, when someone with a high degree of inequality aversion would judge a change to be regressive.

One reason why there is no simple formulae for determining a degree of aversion associated with intersecting profiles is that a single Atkinson measure can arise from a wide range of distributions. Inequality-reducing changes in one range of the distribution can be matched by inequality-increasing changes in another range. Any judge who holds the value judgements implicit in the Atkinson measure will obviously not be worried by the fact that a 'stable' value of inequality is consistent with quite substantial changes in the precise distribution: the judge is by definition indifferent to all such distributions. Hence, the main implications concern the reporting of inequality measures and inferences – particularly policy inferences – drawn from them, when it is recognised that many readers may not share those precise value judgements.

An important lesson is that care must be taken when using and reporting inequality measures. This paper has illustrated a number of ways in which the Atkinson measure can be interpreted, and the associated value judgements explained. Crucially, investigators should not rely on producing a single value, or narrow range, of results: sensitivity analyses are important, even if full details cannot be reported.

Appendix A: Standard Errors of Atkinson's Measure

This appendix gives the properties of sampling distributions for Atkinson's measure. Following Thistle (1990), these are easily computed, but are seldom reported. Here it is necessary to distinguish between population and sample values. Define $y_{p,\varepsilon}$ as the population equally-distributed equivalent income, and corresponding Atkinson inequality measure, $A_{p,\varepsilon}$. Suppose a random sample, $y_1, ..., y_N$, of size n is available. The sample value is treated as a point estimate of $A_{p,\varepsilon}$. This can most conveniently be expressed in terms of fractional sample moments about the origin. Let $\alpha = 1 - \varepsilon$, and let m_α denote the sample moment of order α about the origin. Hence:

$$m_\alpha = \frac{1}{n} \sum_{i=1}^{n} y_i^\alpha \tag{A.1}$$

The sample equally distributed equivalent income is $y_\varepsilon = m_\alpha^{1/\alpha}$, so that the sample Atkinson measure, A_α is:

$$A_\alpha = 1 - \frac{m_\alpha^{1/\alpha}}{m_1} \tag{A.2}$$

where m_1 is the sample arithmetic mean. The asymptotic standard errors can be obtained as follows. Define the higher-order sample moment about the mean, s_α^2, using the standard relationship:

$$s_\alpha^2 = m_{2\alpha} - m_\alpha^2 \tag{A.3}$$

and define the sample covariances using:

$$s_{\alpha_1,\alpha_2} = m_{\alpha_1+\alpha_2} - m_{\alpha_1} m_{\alpha_2} \tag{A.4}$$

Thistle (1990) showed that the asymptotic sampling variance, $V(A_\alpha)$, is given by:

$$V(A_\alpha) = k_\alpha \left[s_\alpha^2 - 2\left(\frac{\alpha m_\alpha}{m_1}\right) s_{\alpha,1} + \left(\frac{\alpha m_\alpha}{m_1}\right)^2 s_1^2 \right] \tag{A.5}$$

where $k_\alpha = \left(\frac{1-A_\alpha}{\alpha m_\alpha}\right)^2$. Furthermore, the asymptotic sampling variance, $V(y_\varepsilon)$, of y_ε can be estimated using:

$$V(y_\varepsilon) = \frac{1}{N} \left(\frac{m_\alpha^{(1-\alpha)/\alpha} s_\alpha}{\alpha} \right)^2 \tag{A.6}$$

These asymptotic variances are therefore easily computed along with the sample values, A_α and y_ε.

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