Energy, momentum, and angular momentum of sound pulses

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Pulse solutions of the wave equation can be expressed as superpositions of scalar monochromatic beam wavefunctions (solutions of the Helmholtz equation). This formulation leads to causal (unidirectional) propagation, in contrast to all currently known closed-form solutions of the wave equation. Application is made to the evaluation of the energy, momentum, and angular momentum of acoustic pulses, as integrals over the beam and pulse weight functions. Equivalence is established between integration over space of the energy, momentum, and angular momentum densities, and integration over the wavevector weight function. The inequality linking the total energy and the total momentum is made explicit in terms of the weight function formulation. It is shown that a general pulse can be viewed as a superposition of phonons, each with energy $\hbar ck$, z component of momentum $\hbar q$, and z component of angular momentum $\hbar m$. A closed-form solution of the wave equation is found, which is localized and causal, and its energy and momentum are evaluated explicitly. https://doi.org/10.1121/1.5014058

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I. INTRODUCTION

This paper is about the basic properties of acoustic pulses in fluids: their energy, momentum, and angular momentum. We shall show that simple universal expressions can be derived for these quantities, and that they relate directly to the phonon picture of sound excitations. We give an explicit pulse wavefunction, forward-propagating everywhere, characterized by a single length parameter a, and thus by a time a/c (c stands for the speed of sound). The complex wavefunction leads to two physical pulses, based on the real and imaginary parts. These have the same energy, momentum, and angular momentum, as we shall prove, but different physical forms. It is hoped that such analytical solutions may be of use in the analysis of problems such as the trapping of spheres by acoustic pulses (Kang and Yeh, 2013) or the generation and observation of acoustical helical wavefronts (Esfahlani *et al.*, 2017). A recent approach to solutions of the wave equation in the space-time domain (Klaseboer *et al.*, 2017) also uses the Helmholtz equation to build scattering solutions of the wave equation.

In linearized hydrodynamics the velocity potential $V(\boldsymbol{r},t)$ of sound pulses satisfies the wave equation, as is well known (Landau and Lifshitz, 1959, Sec. 63)

$$\nabla^2 V(\boldsymbol{r},t) - \partial_{ct}^2 V(\boldsymbol{r},t) = 0, \quad \partial_{ct}^2 V(\boldsymbol{r},t) = c^{-2}\partial_t^2 V(\boldsymbol{r},t). \tag{1}$$

Likewise it is well-known that monochromatic acoustic beams of angular frequency $\omega = ck$ satisfy the Helmholtz equation (which follows on assuming the time dependence e^{-ikct} in solutions of the wave equation)

$$\nabla^2 \mathrm{W}(\boldsymbol{r},k) + k^2 \mathrm{W}(\boldsymbol{r},k) = 0, \quad k = \frac{\omega}{c}. \tag{2}$$

Let $e(\boldsymbol{r},t)$ and $\boldsymbol{p}(\boldsymbol{r},t)$ be the energy and momentum densities, respectively, associated with a sound pulse. The total energy, momentum, and angular momentum are then given by

$$E = \int d^3r\, e(\boldsymbol{r},t), \quad \boldsymbol{P} = \int d^3r\, \boldsymbol{p}(\boldsymbol{r},t),$$
$$\boldsymbol{J} = \int d^3r\, \boldsymbol{r} \times \boldsymbol{p}(\boldsymbol{r},t). \tag{3}$$

In the absence of viscous dissipation and scattering these quantities are all conserved (are independent of time) as may be expected. Explicit demonstration of conservation of energy and momentum and angular momentum are given in Lekner (2006a) and Lekner (2006b), respectively. These quantities are all of second order in the velocity potential when (as is assumed here) the spatial integral of the first-order deviation (due to the pulse) of the density from the density ρ_0 of the undisturbed fluid is zero. This condition, of mass conservation to first order, is equivalent to $\int d^3r\, \partial_{ct} V = 0$. The total energy and z components of the momentum and angular momentum expressed in terms of the velocity potential are then (Landau and Lifshitz, 1959, Sec. 64; Lekner, 2006a)

$$E = \frac{1}{2}\rho_0 \int d^3r \left[(\nabla V)^2 + (\partial_{ct} V)^2\right], \tag{4}$$

$$cP_z = -\rho_0 \int d^3r (\partial_z V)(\partial_{ct} V), \tag{5}$$

$$cJ_z = -\rho_0 \int d^3r (\partial_\phi V)(\partial_{ct} V). \tag{6}$$

We shall be using both Cartesian coordinates $[x,y,z]$ and cylindrical polar coordinates (ρ,ϕ,z), with $\rho = (x^2+y^2)^{1/2}$ the distance from the z axis, and ϕ the azimuthal angle. We have taken the direction of $\boldsymbol{P}$ to coincide with that of positive z.

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The energy of a localized pulse is always greater than the speed of sound times the net momentum, since (as noted in Lekner, 2006a) the difference is proportional to an integral over all space of $(\partial_x V)^2+(\partial_y V)^2+(\partial_z V-\partial_{ct} V)^2$.

As discussed in Lekner (2006b), the component of $\boldsymbol{J}$ parallel to $\boldsymbol{P}$ is invariant to change of origin, and can thus be regarded as the intrinsic angular momentum associated with the pulse. From Eq. (6) we see that acoustic pulses that have a velocity potential independent of the azimuthal angle have zero intrinsic angular momentum, as may be expected.

In Sec. II we shall show how pulses may be viewed as superpositions of beams, and that, for velocity potential independent of the azimuthal angle, this representation is directly related to Bateman's solution of the wave equation in integral form.

II. LOCALIZED SOLUTIONS OF THE WAVE EQUATION

We can construct wavefunctions that have no backward propagating part from the solutions of the wave equation in cylindrical polar coordinates (ρ, ϕ, z), in which it is separable:

$$\left(\partial_\rho^2+\frac{1}{\rho}\partial_\rho+\frac{1}{\rho^2}\partial_\phi^2+\partial_z^2-\partial_{ct}^2\right)J_m(\kappa\rho) \times e^{im\phi}e^{iqz}e^{-ikct}=0 \quad \text{if } \kappa^2+q^2=k^2. \tag{7}$$

$J_m(\kappa\rho)$ is the regular Bessel function of order m. Superposition of such solutions gives the general pulse velocity potential

$$V_m(\rho,\phi,z,t)=e^{im\phi}\int_0^\infty dk f(k)e^{-ikct}\int_0^k d\kappa g(k,\kappa) \times e^{iqz}J_m(\kappa\rho), \quad q=\sqrt{k^2-\kappa^2}. \tag{8}$$

The functions $f(k)$, $g(k,\kappa)$ are arbitrary complex functions, subject only to the existence of the expression (8) and associated integrals, namely, those which give the total energy, momentum, and angular momentum of a pulse. A more compact form is obtained by conflating the functions $f(k)$, $g(k,\kappa)$ into their product $f(k,\kappa)$,

$$V_m(\rho,\phi,z,t)=e^{im\phi}\int_0^\infty dk e^{-ikct}\int_0^k d\kappa f(k,\kappa) \times e^{iqz}J_m(\kappa\rho), \quad q=\sqrt{k^2-\kappa^2}. \tag{9}$$

The form of the Bessel-based pulses in Eq. (8) or (9) connects them to *generalized Bessel beams* (Lekner, 2006c, 2007). For integral m we define a scalar monochromatic beam of angular frequency $\omega=kc$ by

$$W_m(\rho,\phi,z,\,k)=e^{im\phi}\int_0^k d\kappa f(k,\kappa)e^{iqz}J_m(\kappa\rho). \tag{10}$$

Such beam wavefunctions automatically satisfy the Helmholtz equation $(\nabla^2+k^2)W(\boldsymbol{r},\,k)=0$. The pulse defined in Eq. (9) is seen to be a superposition of generalized Bessel beams

$$V_m(\rho,\phi,z,t)=\int_0^\infty dk e^{-ikct}W_m(\rho,z,\,k). \tag{11}$$

Bateman (1904) obtained a *general solution of the wave equation in integral form*. For solutions with axial symmetry (independent of the azimuthal angle ϕ) the nonsingular part of the Bateman solution is, with $F(u,v)$ an arbitrary twice-differentiable function,

$$V(\rho,z,t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta F(z+i\rho\cos\theta, ct+\rho\sin\theta)\,. \tag{12}$$

The general solution includes a term that is logarithmically singular on the axis $\rho=0$, which we omit. A proof (different from Bateman's) that Eq. (12) satisfies the wave equation is given in Lekner (2016a, p. 491).

On the propagation axis $\rho=0$ the pulse wavefunction [Eq. (12)] becomes $V(0,z,t)=F(z,t)$. For example, if the on-axis wavefunction takes the form $e^{ik(z-ct)}$, Bateman's integral becomes

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\, e^{ik\left(z-ct+i\rho e^{i\theta}\right)}=\frac{e^{ik(z-ct)}}{2\pi i}\oint du\frac{e^{-k\rho u}}{u}=e^{ik(z-ct)}. \tag{13}$$

($u=e^{i\theta}$ and the second integral is around the unit circle, with a simple pole at $u=0$.) Thus, the plane-wave form on the axis determines the whole pulse to be a plane wave.

Bateman's integral solution [Eq. (12)] is directly related to the $m=0$ form of Eq. (9). On the z axis we have

$$V_0(0,z,t)=\int_0^\infty dk e^{-ikct}\int_0^k d\kappa f(k,\kappa)e^{iqz}=F(z,t). \tag{14}$$

The full wavefunction is thus, on substitution into Eq. (12),

$$V_0(\rho,z,t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\int_0^\infty dk e^{-ik(ct+\rho\sin\,\theta)} \times\int_0^k d\kappa f(k,\kappa)e^{iq(z+i\rho\cos\,\theta)}. \tag{15}$$

By Bessel's integral (Watson, 1944, Sec. 2.21) the integration over θ gives $2\pi J_0(\kappa\rho)$, and we regain the $m=0$ form of Eq. (9),

$$V_0(\rho,z,t)=\int_0^\infty dk e^{-ikct}\int_0^k d\kappa f(k,\kappa)e^{iqz}J_0(\kappa\rho). \tag{16}$$

We return to the *representation of pulses as superposition of beams*: if $W(\boldsymbol{r},k)$ satisfies the Helmholtz equation, a set of pulse velocity potentials satisfying the wave equation can be obtained by integration over wavenumber (with some weight function of k incorporated into W),

$$V(\boldsymbol{r},t)=\int_0^\infty dk e^{-ikct}W(\boldsymbol{r},k). \tag{17}$$

As the simplest example, if the beam wavefunction is a plane wave, $W(\boldsymbol{r},k)=G(k)e^{ikz}$, we obtain the general plane-wave pulse $V(z-ct)$, extending to infinity in the x and y directions.

To obtain a pulse localized in space-time we need to start with a beam that is transversely localized. A closed-form example of a transversely localized beam is provided by a simple beam wavefunction (Lekner, 2001),

$$W(\boldsymbol{r},k) = e^{-ka'}R^{-1}\sin kR \quad (a'>0),$$
$$R^2 = \rho^2 + (z-ib')^2. \tag{18}$$

Equation (17) then gives us

$$V(\boldsymbol{r},t) = \frac{1}{2iR}\int_0^\infty dk e^{-ka'}e^{-ikct}\left(e^{ikR}-e^{-ikR}\right)$$
$$= \frac{1}{R^2+(a'+ict)^2}. \tag{19}$$

We recognize this wavefunction as a variant of the solution of the wave equation used in Lekner (2006a). To obtain an exact correspondence, we set

$$a' = (a+b)/2, \quad b' = -(a-b)/2. \tag{20}$$

Then Eq. (19) reduces to

$$V(\boldsymbol{r},t) = \frac{1}{\rho^2+[a-i(z+ct)][b+i(z-ct)]}$$
$$= \frac{1}{r^2-(ct)^2+ab+i[(a-b)z-(a+b)ct]}. \tag{21}$$

There is a more direct way of obtaining this wavefunction: The function $(x^2+y^2+z^2-c^2t^2)^{-1} = (r^2-c^2t^2)^{-1}$ solves the wave equation, but is singular on the "sound-cone" $r^2=c^2t^2$. Complex displacements in space or time make it nonsingular at real space-time points, an idea credited by Trautman (1962) to Synge (1960). The complex displacements $z\to z+i(a-b)/2$, $ct\to ct+i(a+b)/2$ result in the wavefunction [Eq. (21)]. Ziolkowski (1985) discusses cylindrically symmetric solutions of the wave equation of the form

$$\psi_Z(\rho,z,t) = [b+i(z-ct)]^{-1}$$
$$\times\int_0^\infty dk\,F(k)e^{ik(z+ct)-k\rho^2/[b+i(z-ct)]}. \tag{22}$$

The choice $F(k)=abe^{-ka}$ reproduces the simple wavefunction [Eq. (21)].

Hillion (1993) notes that the wave equation is solved by functions of the form

$$\psi_H(\rho,z,t) = \frac{f(s)}{b+i(z-ct)}, \quad s=\frac{\rho^2}{b+i(z-ct)}-i(z+ct). \tag{23}$$

With $f(s)=ab/(s+a)$ we regain the wavefunction [Eq. (21)]. An oscillatory wavefunction results from the choice $f(s)=abe^{-ks}/(s+a)$ (Lekner, 2006d)

$$\psi(\rho,z,t) = \frac{ab\,e^{ik(z+ct)-k\rho^2/b+i(z-ct)}}{\rho^2+[a-i(z+ct)][b+i(z-ct)]}. \tag{24}$$

The Hillion form of solutions given in Eq. (24) can be generalized to give pulses with *azimuthal dependence* (Lekner, 2006b). One can verify that the following are wavefunctions:

$$\frac{g}{b+i(z-ct)}\psi_H \quad (g=x,\,y,\,z,\,x\pm iy),$$
$$\frac{h}{[b+i(z-ct)]^2}\psi_H \quad \left(h=xy,\,x^2-y^2,\,(x\pm iy)^2\right). \tag{25}$$

Of particular interest are wavefunctions with azimuthal dependence $e^{im\phi}$. We note that $x+iy=\rho e^{i\phi}$, and can verify that the following are wavefunctions for any twice-differentiable f:

$$\left[\frac{\rho}{b+i(z-ct)}\right]^{|m|}e^{im\phi}\frac{f(s)}{b+i(z-ct)},$$
$$s=\frac{\rho^2}{b+i(z-ct)}-i(z+ct). \tag{26}$$

The values of m are restricted to integers by the condition $\psi(\rho,\phi+2\pi,z)=\psi(\rho,\phi,z)$.

It is notable that all of these closed-form solutions of the wave equation contain both $z-ct$ and $z+ct$. (More examples may be found in Sec. 4 of the review by Kiselev, 2007.) They cannot originate from a single radiation source or a localized array of sources. On the other hand, the form of Eqs. (8) and (9) guarantees the absence of $z+ct$ terms: The integrand contains the factor $e^{i(qz-kct)}$, with $k\ge q\ge 0$. We shall assume a wavefunction of the form of Eq. (9) and calculate the resultant energy, momentum, and angular momentum in terms of the weight function $f(k,\kappa)$ in Sec. III. An explicit solution of the wave equation will be given in Sec. IV, and its properties will be discussed in detail.

III. CALCULATION OF THE ENERGY, MOMENTUM, AND ANGULAR MOMENTUM

The most convenient form of the general solution of the wave equation given in Eq. (9) is complex with real and imaginary parts V_r, V_i of the velocity potential: $V=V_r+iV_i$. Suppose we use the real part $V_r=(1/2)(V+V^*)$, and wish to calculate the integral over all space of $(\partial_z V_r)(\partial_{ct}V_r)=(1/4)(\partial_z V+\partial_z V^*)(\partial_{ct}V+\partial_{ct}V^*)$. As we shall see below, the integration over z gives rise to delta functions, which in the case of the products $(\partial_z V)(\partial_{ct}V)$ and $(\partial_z V^*)(\partial_{ct}V^*)$ are both $\delta(q+q')$. Since both q and q' are by definition non-negative, these terms integrate to zero. (When $m\neq 0$ this is to be multiplied by another zero, from the integration of $e^{\pm 2im\phi}$ over ϕ.) Only the terms $(\partial_z V^*)(\partial_{ct}V)$ and $(\partial_z V)(\partial_{ct}V^*)$ remain. The same is true if we choose to use the imaginary part $V_i=(1/2i)(V-V^*)$, and, in fact, the total energy, momentum, and angular momentum are the same whether the velocity potential is chosen as the real or the imaginary part of the complex potential [Eq. (9)]. Therefore the energy, momentum, and angular momentum expressed in terms of the complex velocity potential are

$$E=\frac{1}{2}\rho_0\int d^3r\left[|\nabla V|^2+|\partial_{ct}V|^2\right], \tag{27}$$

$$cP_z=-\frac{1}{2}\rho_0\int d^3r\,\mathrm{Re}\{(\partial_z V^*)(\partial_{ct}V)\}, \tag{28}$$

$$cJ_z = -\frac{1}{2}\rho_0 \int d^3r \, \mathrm{Re}\{(\partial_\phi V^*)(\partial_{ct} V)\}. \tag{29}$$

The fact that the real and imaginary parts V_r, V_i give the same values of the total energy, momentum, and angular momentum (although the respective densities are different) is not too surprising, given the arbitrariness of the real/imaginary division: If we multiply the complex velocity potential by a complex number of unit modulus ($e^{i\theta}$, say), the norm $N = \int d^3r \, |V|^2$ remains the same, while the real and imaginary parts transform into linear combinations of their original values

$$V_r \to V_r \cos\theta - V_i \sin\theta, \quad V_i \to V_i \cos\theta + V_r \sin\theta. \tag{30}$$

In the evaluation of the total energy, momentum, and angular momentum we shall use the techniques introduced in the calculation of the invariants of acoustic beams (Lekner, 2006c, 2007). Let us evaluate the *angular momentum* first, for which we need the spatial integral

$$\int d^3r \, (\partial_\phi V^*)(\partial_{ct} V) = \int_0^\infty d\rho\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \, (\partial_\phi V^*)(\partial_{ct} V). \tag{31}$$

The differentiation of V^* with respect to ϕ brings down a factor of $-im$, differentiation of V with respect to ct gives a factor of $-ik'$. The integration over ϕ gives a factor of 2π. Thus

$$\begin{aligned}\int d^3r \, (\partial_\phi V^*)(\partial_{ct} V) &= -2m\pi \int_0^\infty d\rho\rho \int_{-\infty}^{\infty} dz \int_0^\infty dk k e^{ikct} \\ &\times \int_0^\infty dk' k' e^{-ik'ct} \\ &\times \int_0^k d\kappa \kappa^{-1} f(k,\kappa)^* e^{-iqz} J_m(\kappa\rho) \\ &\times \int_0^{k'} d\kappa' f(k',\kappa') e^{iq'z} J_m(\kappa'\rho).\end{aligned} \tag{32}$$

In carrying out the integration over ρ we shall use Hankel's inversion formula (Watson, 1944, Sec. 14.4), which in modern notation reads

$$\int_0^\infty d\rho\rho J_m(\kappa\rho) J_m(\kappa'\rho) = \kappa^{-1}\delta(\kappa - \kappa') \quad (\kappa, \kappa' > 0). \tag{33}$$

Thus

$$\begin{aligned}\int d^3r \, (\partial_\phi V^*)(\partial_{ct} V) &= -2\pi m \int_{-\infty}^{\infty} dz \int_0^\infty dk \, k e^{ikct} \\ &\times \int_0^\infty dk' \, k' e^{-ik'ct} \int_0^k d\kappa \, \kappa^{-1} f(k,\kappa)^* \\ &\times f(k',\kappa) \, e^{iz(\sqrt{k'^2-\kappa^2} - \sqrt{k^2-\kappa^2})}.\end{aligned} \tag{34}$$

Next we perform the z integration, and use the Fourier inversion formula

$$\begin{aligned}\int_{-\infty}^{\infty} dz \, e^{iz(q'-q)} &= 2\pi \, \delta(q' - q) \\ &= 2\pi \, \delta\left(\sqrt{k'^2 - \kappa^2} - \sqrt{k^2 - \kappa^2}\right) \\ &= 2\pi \, \frac{\sqrt{k^2 - \kappa^2}}{k} \delta(k' - k).\end{aligned} \tag{35}$$

The last equality follows from the relation [which assumes $G(k)$ to be monotonic in k, so $G(k') = G(k)$ when $k' = k$ and nowhere else]

$$\delta(G(k') - G(k)) = \left|\frac{dk}{dG}\right| \delta(k' - k). \tag{36}$$

Thus, the spatial integrations have removed two of the wave-number integrations and we are left with

$$cJ_z = \frac{1}{2}\rho_0 m (2\pi)^2 \int_0^\infty dk \int_0^k d\kappa |f(k,\kappa)|^2 \kappa^{-1} \sqrt{k^2 - \kappa^2}. \tag{37}$$

An interesting aspect of this result is that the integrals do not depend on the azimuthal index m, but only on the absolute square of the weight function $f(k,\kappa)$. If we take $f(k,\kappa)$ to be independent of m, J_z is strictly proportional to m, and carries the sign of m, since the integrand is non-negative.

In the *momentum* calculation, differentiation of V^* with respect to z brings down a factor of $-iq$, differentiation of V with respect to ct gives a factor of $-ik'$. Proceeding as above, the spatial integrations give

$$cP_z = \frac{1}{2}\rho_0 (2\pi)^2 \int_0^\infty dk \int_0^k d\kappa |f(k,\kappa)|^2 \kappa^{-1} (k^2 - \kappa^2). \tag{38}$$

Again, if we take $f(k,\kappa)$ to be independent of m, the integrals do not depend on the azimuthal index, and again the integrand is non-negative. For the acoustic velocity potential as defined in Eq. (9), the momentum component along the z axis is positive.

We come finally to the *energy* integral. The time derivative part is straightforward. Differentiation of V^* with respect to ct brings down a factor of ik, and differentiation of V with respect to ct gives a factor of $-ik'$. The spatial integrations can be carried out as before with the result

$$\begin{aligned}\int d^3r (\partial_{ct} V^*)(\partial_{ct} V) &= (2\pi)^2 \int_0^\infty dk k \\ &\times \int_0^k d\kappa |f(k,\kappa)|^2 \kappa^{-1} \sqrt{k^2 - \kappa^2}.\end{aligned} \tag{39}$$

Next, we must evaluate the spatial integrals over

$$\begin{aligned}(\nabla V^*)(\nabla V) &= (\partial_\rho V^*)(\partial_\rho V) + \rho^{-2}(\partial_\phi V^*)(\partial_\phi V) \\ &+ (\partial_z V^*)(\partial_z V).\end{aligned} \tag{40}$$

Differentiation of V^* with respect to z brings down a factor of $-iq$, differentiation of V with respect to z gives a factor of iq'. Hence,

$$\begin{aligned}\int d^3r \, (\partial_z V^*)(\partial_z V) &= (2\pi)^2 \int_0^\infty dk \\ &\times \int_0^k d\kappa |f(k,\kappa)|^2 \kappa^{-1} (k^2 - \kappa^2).\end{aligned} \tag{41}$$

The remaining terms, $(\partial_\rho V^*)(\partial_\rho V) + \rho^{-2}(\partial_\phi V^*)(\partial_\phi V)$, have the Bessel function parts

$$\partial_\rho J_m(\kappa\rho) \, \partial_\rho J_m(\kappa'\rho) + m^2 \rho^{-2} J_m(\kappa\rho) J_m(\kappa'\rho). \tag{42}$$

We use the identities (Olver and Maximon, 2010, relations 10.6.1)

$$2J'_m(\zeta) = J_{m-1}(\zeta) - J_{m+1}(\zeta),$$
$$\frac{2m}{\zeta}J_m(\zeta) = J_{m-1}(\zeta) + J_{m+1}(\zeta). \tag{43}$$

These relations reduce the expression (42) to a form amenable to the Hankel inversion formula, namely, to

$$\frac{\kappa\kappa'}{2}\{J_{m-1}(\kappa\rho)J_{m-1}(\kappa'\rho) + J_{m+1}(\kappa\rho)J_{m+1}(\kappa'\rho)\}. \tag{44}$$

The spatial integrations can now be performed as before,

$$\int d^3r\left[(\partial_\rho V^*)(\partial_\rho V) + \rho^{-2}(\partial_\phi V^*)(\partial_\phi V)\right]$$
$$= (2\pi)^2\int_0^\infty dk\,k^{-1}\int_0^k d\kappa\,|f(k,\kappa)|^2\kappa\sqrt{k^2-\kappa^2}. \tag{45}$$

Thus the total energy is given by

$$E = \frac{1}{2}\rho_0(2\pi)^2\int_0^\infty dk\int_0^k d\kappa|f(k,\kappa)|^2 k\kappa^{-1}\sqrt{k^2-\kappa^2}. \tag{46}$$

The pulse energy again does not depend on the azimuthal winding number m [always assuming that the weight function $f(k,\kappa)$ is independent of m]. The pulse energy is positive definite, and further, as we noted in the Introduction, the pulse energy always exceeds the net momentum times the speed of sound since

$$E - cP_z = \frac{1}{4}\rho_0(2\pi)^2\int_0^\infty dk\,k^{-1}$$
$$\times\int_0^k d\kappa|f(k,\kappa)|^2\kappa^{-1}\sqrt{k^2-\kappa^2}\{k - \sqrt{k^2-\kappa^2}\}. \tag{47}$$

To summarize the results so far obtained (with $q = \sqrt{k^2-\kappa^2}$),

$$\begin{bmatrix} E \\ cP_z \\ cJ_z \end{bmatrix} = \pi^2\rho_0\int_0^\infty dk\int_0^k d\kappa|f(k,\kappa)|^2\kappa^{-1}q\begin{bmatrix} k \\ q \\ m \end{bmatrix}. \tag{48}$$

For comparison, the norm of the velocity potential (calculated by the same methods) is

$$N \equiv \int d^3r\,|V|^2 = (2\pi)^2\int_0^\infty dk\,k^{-1}\int_0^k d\kappa\,|f(k,\kappa)|^2\kappa^{-1}q. \tag{49}$$

Again there is no dependence on the azimuthal index m when $f(k,\kappa)$ is independent of m. Incidentally, we have proved that E, P_z, J_z, and N are all constant in time.

An equivalent representation is in terms of integration over the wavenumber q, instead of over κ. We have $k^2 = \kappa^2 + q^2$, $\kappa d\kappa + q dq = 0$, and with $h(k,q) = q\kappa^{-1} f(k,\kappa)$ the normalization integral is

$$N = (2\pi)^2\int_0^\infty dk\,k^{-1}\int_0^k dq\,|h(k,q)|^2. \tag{50}$$

The pulse energy, momentum, and angular momentum are also somewhat simpler, and the expression for the z component of momentum has a more physical appeal, being proportional to the integral over q (the longitudinal wavenumber) times $|h(k,q)|^2$,

$$\begin{bmatrix} E \\ cP_z \\ cJ_z \end{bmatrix} = \pi^2\rho_0\int_0^\infty dk\int_0^k dq|h(k,q)|^2\begin{bmatrix} k \\ q \\ m \end{bmatrix}. \tag{51}$$

We note that the pulse energy is always greater than the net pulse momentum, as expected, since $k \geq q$.

IV. AN ACOUSTIC PULSE BASED ON THE PROTO-BEAM

The proto-beam is the confluent tight-focus limit of two families of beams, transversely bounded exact solutions of the Helmholtz equation. The first family was introduced by Carter (1973) and its properties were explored by Berry (1998) and Nye (1998),

$$W_C(\rho,z) = \int_0^k dq\,q e^{bq^2/2k+iqz}J_0(\rho\sqrt{k^2-q^2}). \tag{52}$$

The second was considered in Lekner (2016b),

$$W_b(\rho,z) = \int_0^k dq\,q e^{qb+iqz}J_0(\rho\sqrt{k^2-q^2}). \tag{53}$$

The lengths k^{-1} and b determine the extent of the focal region. When kb is large the longitudinal extent is of order b and the transverse extent is of order $\sqrt{b/k}$. As $b \to 0$ the only length remaining is k^{-1}, and both longitudinal and transverse extents of the focal region are of order k^{-1}. This is the case for the confluent limit of W_C, W_b as b tends to zero

$$W_0(\rho,z,k) = \int_0^k dq q\,e^{iqz}\,J_0(\rho\sqrt{k^2-q^2}). \tag{54}$$

Explicit expressions for W_0 are given in Lekner (2016b) in terms of Lommel functions of two variables, and alternatively in terms of a series expansion in spherical Bessel functions and Legendre polynomials. We shall construct a solution of the wave equation from a superposition of $W_0(\rho,z,k)$ with the weight functions $f(k,\kappa)$ or $h(k,q)$ given by

$$f(k,\kappa) = \frac{a^4}{3}ke^{-ka}\kappa,\quad h(k,q) = q\kappa^{-1}f(k,\kappa) = \frac{a^4}{3}ke^{-ka}q. \tag{55}$$

The resulting solution of the wave equation is

$$V_0(\rho,z,t) = \frac{a^4}{3}\int_0^\infty dk\,ke^{-ka-ikct}\int_0^k dq\,qe^{iqz}J_0\left(\rho\sqrt{k^2-q^2}\right). \tag{56}$$

The prefactor in Eqs. (55) and (56) has been chosen to make the velocity potential unity at the space-time origin: $V_0(0,0,0) = 1$. To evaluate Eq. (56) we first use Bessel's integral (Watson, 1944, Sec. 2.21) to write

$$J_0\left(\rho\sqrt{k^2-q^2}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\,e^{\rho(q\cos\theta + ik\sin\theta)}. \tag{57}$$

The order of integrations over q and then over k may be inverted, since it is assumed that $a > 0$. In the final integration over θ an ambiguity arises in the term $\ln(-a-\rho+iz)$, which we interpret as $-i\pi + \ln(a+\rho-iz)$. The result is

$$V_0(\rho,z,t) = \frac{a^4\tilde{a}}{3}\frac{3\left(\tilde{a}^2+\rho^2\right)^2 - 6z^2\left(\tilde{a}^2+\rho^2\right) - z^4 + 8iz\left(\tilde{a}^2+\rho^2\right)^{3/2}}{\left(\tilde{a}^2+\rho^2\right)^{3/2}\left(\tilde{a}^2+\rho^2+z^2\right)^3}, \qquad \tilde{a} = a + ict. \tag{58}$$

Differentiation verifies that V_0 satisfies the wave equation. At $t=0$ the absolute square of V_0 is simple

$$|V_0(\rho,z,0)|^2 = \frac{a^{10}\left(9a^2+9\rho^2+z^2\right)}{9(a^2+\rho^2)^3(a^2+\rho^2+z^2)^3}. \tag{59}$$

We note that the lateral decay of $|V_0(\rho,z,0)|$ is faster than the longitudinal decay: Asymptotically these are ρ^{-5} and z^{-2}, respectively. On the propagation axis ($\rho=0$) the modulus squared is

$$|V_0(0,z,t)|^2 = \frac{a^8\left[9a^2+(z-3ct)^2\right]}{9(a^2+c^2t^2)^2\left[a^2+(z-ct)^2\right]^3}. \tag{60}$$

Propagation is in the positive z direction. There is no backward propagation part containing $z+ct$ as we had in the solutions (21)–(25). Figure 1 shows the time development of the modulus of V_0.

The norm may now be calculated directly,

$$\begin{aligned} N &= \int d^3r|V_0(\rho,z,0)|^2 \\ &= 2\pi\int_0^\infty d\rho\rho\int_{-\infty}^{\infty}dz\,|V_0(\rho,z,0)|^2 = \frac{\pi^2}{9}a^3. \end{aligned} \tag{61}$$

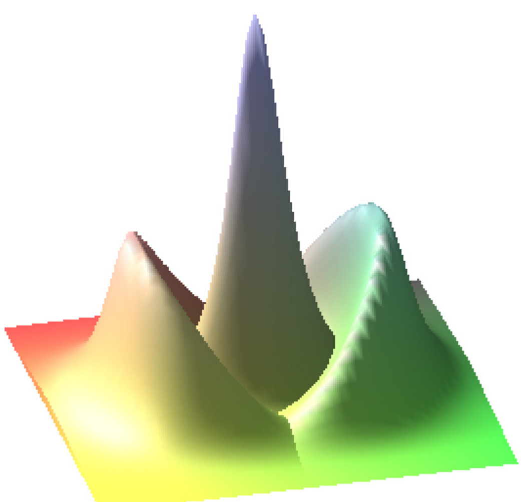

FIG. 1. (Color online) Plots of the modulus $|V_0(\rho,z,t)|$ at times $t=0$, $ct=\pm 3a$. The central $t=0$ peak has been scaled down by a factor of two so that it does not swamp the neighboring moduli at $ct=\pm 3a$. Propagation is from left to right. There is symmetry about the space-time origin, which is the center of the focal region of the pulse.

Equivalently we may use the expression (50),

$$\begin{aligned} N &= (2\pi)^2\int_0^\infty dk\,k^{-1}\int_0^k dq|h(k,q)|^2 \\ &= \frac{4\pi^2}{9}a^8\int_0^\infty dkk\,e^{-2ka}\int_0^k dqq^2 = \frac{\pi^2}{9}a^3. \end{aligned} \tag{62}$$

For given length parameter a the effective volume of the pulse may be represented as a sphere of radius $(\pi/12)^{1/3}a \approx 0.64a$.

We turn now to the calculation of the total energy and total momentum of the pulse. (The angular momentum is zero, since there is no azimuthal dependence in the velocity potential.) The energy and momentum densities are (see Landau and Lifshitz, 1959, Sec. 64; Lekner, 2006a, Sec. 3)

$$\begin{aligned} e(\boldsymbol{r},t) &= \frac{1}{2}\rho_0\left[(\nabla V)^2 + (\partial_{ct}V)^2\right], \\ \boldsymbol{p}(\boldsymbol{r},t) &= -c^{-1}\rho_0(\partial_{ct}V)\nabla V. \end{aligned} \tag{63}$$

In cylindrical coordinates the two nonzero components of the momentum density are given by

$$cp_\rho = -\rho_0(\partial_{ct}V)\partial_\rho V, \quad cp_z = -\rho_0(\partial_{ct}V)\partial_z V. \tag{64}$$

Although the total energy and momentum of a pulse are the same for the real and for the imaginary parts of a complex velocity potential $V = V_r + iV_i$, the energy and momentum densities are different. We shall look at those derived from the *real part* of Eq. (58) first. We see from Fig. 2 that the energy and momentum densities derived from V_r are zero at the space-time origin, with symmetrically spaced peaks on either side, the energy density being maximum at $z \approx \pm 0.3565a$ (this number is the real root of a quintic in z, which comes from the differentiation of the on-axis energy density). As time increases the two maxima merge and asymptotically form a single peak traveling at speed c in the z direction, while spreading sideways (Fig. 3).

Next we look at those derived from the *imaginary part* of Eq. (58). Figures 4 and 5 show that the energy and momentum densities derived from V_i are maximal at the space-time origin. As time increases the pulse splits into two parts, and asymptotically forms a pair of energy density peaks traveling at speed c along the axis. Far from the origin (which is the center of the focal region) the minimum between the peaks is at $z=ct$, and the energy density peaks are located at $z = ct \pm a\sqrt{1-2/\sqrt{5}} \approx ct \pm 0.3249a$. Their separation is thus asymptotically about $0.65a$.

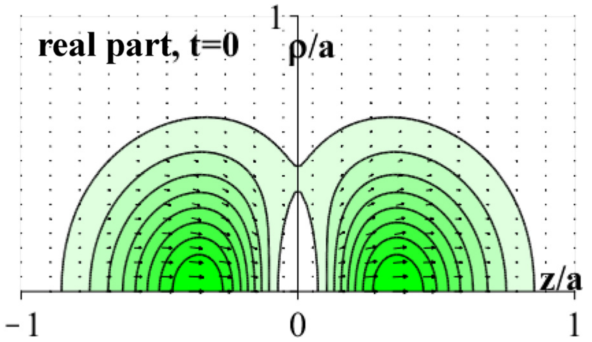

FIG. 2. (Color online) Energy and momentum densities at $t = 0$ corresponding to the real part V_r of the velocity potential [Eq. (58)]. The energy density is shown by shading and contours and the momentum density is shown by arrows. Both are zero at the origin.

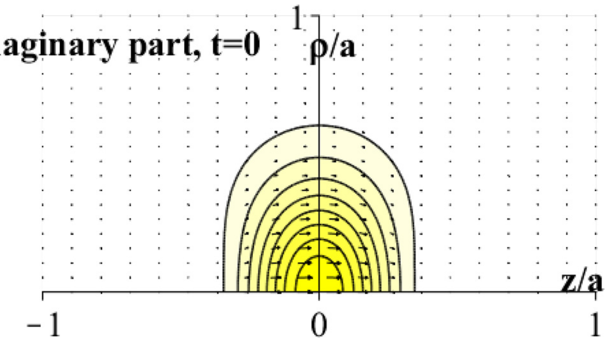

FIG. 4. (Color online) Energy and momentum densities at $t = 0$ corresponding to the imaginary part V_i of the velocity potential [Eq. (58)]. As in Fig. 2 the energy density is shown by shading and contours and the momentum density is shown by arrows. Both are maximal at the origin.

The total energy and total momentum of the acoustic pulse based on $V_0(\rho, z, t)$ may be found from Eqs. (48) or (51), with the functions f, h being given by the expressions in Eq. (55) multiplied by $\bar{V}$, the magnitude of the complex velocity potential at the space-time origin (up until now we have normalized the velocity potential at the space-time origin to unity, but of course it depends on the amplitude of the pulse). The results are

$$E = \frac{5}{6}\pi^2 \rho_0 a \bar{V}^2, \quad cP_z = \frac{5}{16}\pi^2 \rho_0 a \bar{V}^2. \tag{65}$$

The same results follow for both the real and the imaginary parts from evaluation of the space integrals in Eqs. (4)–(6), and from the integration in Eq. (27) using complex V. The pulses based on the real and imaginary parts converge toward the focal region and then diverge from it, as we saw analytically and graphically. Because the convergence and divergence are strong, the ratio of speed of sound times net momentum to the energy is substantially less than unity, namely, $3/8$.

V. DISCUSSION

We have seen that the energy, momentum, and angular momentum of an acoustic pulse may be expressed as integrals over a wavenumber weight function. We showed in Sec. III that the real and imaginary parts of the complex velocity potential, although giving distinct pulses, have the same energy, momentum, and angular momentum, all independent of time in the absence of dissipation and scattering. The results of Sec. III are summarized in the three formulas

$$\begin{bmatrix} E \\ cP_z \\ cJ_z \end{bmatrix} = \pi^2 \rho_0 \int_0^\infty dk \int_0^k dq |h(k,q)|^2 \begin{bmatrix} k \\ q \\ m \end{bmatrix}. \tag{66}$$

These equations have a simple interpretation in terms of phonons: The pulse can be viewed as a superposition of phonons, each with energy $\hbar ck$, z component of momentum $\hbar q$, and z component of angular momentum $\hbar m$.

It is interesting to compare Eq. (66) with the expressions for the energy, momentum, and angular momentum *per unit length* of acoustic beams derived from the general beam wavefunction [EQ. (10)], namely,

$$W_m(\rho, \phi, z, k) = e^{im\phi} \int_0^k d\kappa f(k,\kappa) e^{iqz} J_m(\kappa\rho),$$
$$q = \sqrt{k^2 - \kappa^2}. \tag{67}$$

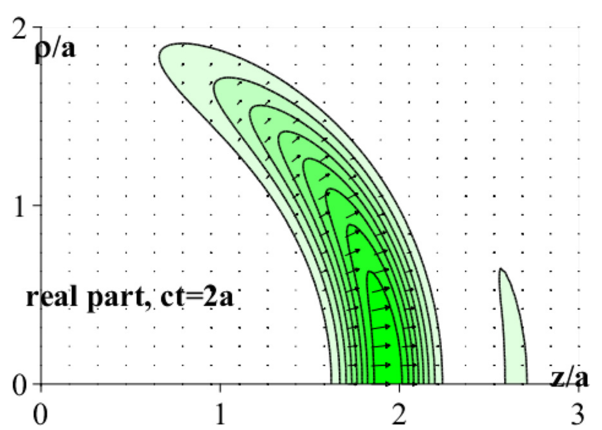

FIG. 3. (Color online) Energy and momentum densities at $ct = 2a$ corresponding to the real part V_r of the velocity potential [Eq. (58)]. Notation as in Fig. 2. The three-dimensional picture is obtained by rotation about the direction of propagation (the horizontal axis).

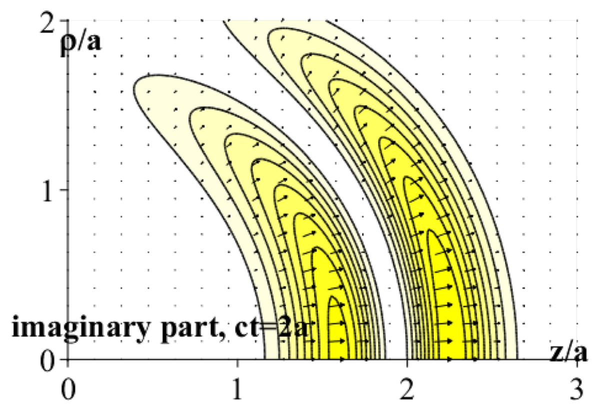

FIG. 5. (Color online) Energy and momentum densities at $ct = 2a$ corresponding to the imaginary part V_i of the velocity potential [Eq. (58)]. Away from the focal region the two-peak structure persists, asymptotically with a fixed separation between the energy density maxima, as discussed in the text.

We denote by E' the energy per unit length, and likewise for P'_z, J'_z. The results found in Lekner (2006c, 2007) are

$$\begin{bmatrix} E' \\ cP'_z \\ cJ'_z \end{bmatrix} = \pi\rho_0 k \int_0^k d\kappa\, \kappa^{-1} |f(k,\kappa)|^2 \begin{bmatrix} k \\ q \\ m \end{bmatrix}. \tag{68}$$

Again, the acoustic beam can be viewed as a superposition of phonons, each with energy $\hbar ck$, z component of momentum $\hbar q$, and z component of angular momentum $\hbar m$. The (idealized) beam is a spatially static entity, while the pulse is inherently dynamic and changes position and shape as time evolves. In both cases there are conserved quantities: The pulse energy, momentum, and angular momentum are independent of time, and the beam energy, momentum, and angular momentum per unit length are independent of position along the beam. (The last statement is for generalized Bessel beams; for discussion of the set of acoustic beam invariants, see Lekner, 2006c, 2007.)

We note that he concept of axial angular momentum carried by acoustic beams having an $e^{im\phi}$ azimuthal dependence was originally discussed by Hefner and Marston (1999) in a paper describing the experimental synthesis of such beams, although the analysis of angular momentum was limited to paraxial beams. The paraxial limitations in Hefner and Marston (1999) were subsequently removed by the analysis in Zhang and Marston (2011), and experiments confirmed the predicted scaling of acoustic radiation torque in proportion to m. The result [Eq. (68)] for monochromatic beams gives $ckJ'_z = mE'$, or $\omega J'_z = mE'$. The same ratio follows by integration of the flux density ratio, Eq. (5) of Zhang and Marston (2011). In experiments such as those of Démoré *et al.* (2012), the flux ratio in Eq. (5) of Zhang and Marston (2011) becomes directly relevant because experiments are often limited by the power available as opposed to the total beam energy per unit length E' calculated in Lekner (2006c, 2007). We note also that analysis of wave fields with non-zero m has been shown to be useful for understanding situations where small symmetric objects are placed on the axis (Zhang and Marston, 2014). In such situations the inner wave-field region is more important than the outer region when modeling the response of the small inserted object.

To date, all of the known localized closed-form solutions of the wave equation have had some backward propagation in them, as seen in Sec. II. In Sec. IV we gave a derivation of a strictly causal complex pulse, with only forward propagation. The pulse is characterized by one length, a. The energy and momentum of this pulse were evaluated (the angular momentum is zero). Graphs and analytics of the pair of pulses derived from the real and the imaginary parts of the velocity potential show strong convergence/divergence, with a tight focal region of size a.

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