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Electromagnetic pulses, localized and causal

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We show that pulse solutions of the wave equation can be expressed as time Fourier superpositions of scalar monochromatic beam wave functions (solutions of the Helmholtz equation). This formulation is shown to be equivalent to Bateman's integral expression for solutions of the wave equation, for axially symmetric solutions. A closed-form one-parameter solution of the wave equation, containing no backward-propagating parts, is constructed from a beam which is the tight-focus limit of two families of beams. Application is made to transverse electric and transverse magnetic pulses, with evaluation of the energy, momentum and angular momentum for a pulse based on the general localized and causal form. Such pulses can be represented as superpositions of photons. Explicit total energy and total momentum values are given for the one-parameter closed-form pulse.

1. Introduction

Recent work has shown that, for *any* sound pulse in a homogeneous fluid which is based on the general localized and causal form of solutions of the wave equation, the energy, momentum and angular momentum can be evaluated in terms of an integral over the weight function which defines the solution [1]. This reduction is useful because the integrals over the weight function are simpler to evaluate and more physically meaningful than the corresponding integration of energy, momentum and angular momentum densities over all of space. It also gives a representation of the sound pulse as a superposition of phonons.

A similar reduction is possible for electromagnetic pulses of specific type. In this paper we shall show that for transverse electric (TE) and transverse magnetic

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(TM) pulses the total energy, momentum and angular momentum are given by

$$\begin{bmatrix} U \\ cP_z \\ cJ_z \end{bmatrix} = \frac{\pi}{2}\int_0^\infty \mathrm{d}k \int_0^k \mathrm{d}\kappa\, |f(k,\kappa)|^2 \kappa q \begin{bmatrix} k \\ q \\ m \end{bmatrix}, \quad q = \sqrt{k^2-\kappa^2}, \tag{1.1}$$

where U, P_z and J_z are, respectively, the total energy, the z component of the momentum and the z component of the angular momentum. For pulses propagating in the z-direction, and converging towards or diverging from the propagation axis (as all localized pulses must), the transverse components of momentum are zero, and J_z is the only component of angular momentum invariant to choice of origin. In (1.1), q and κ are the longitudinal and transverse components of the wavevector, respectively, and c is the speed of light. The function $f(k,\kappa)$ is the wavenumber weight function which defines the solution of the wave equation on which the electromagnetic pulse is based, as discussed in §2:

$$\psi_m(\rho,\phi,z,t) = \mathrm{e}^{\mathrm{i}m\phi}\int_0^\infty \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kct}\int_0^k \mathrm{d}\kappa f(k,\kappa)\mathrm{e}^{\mathrm{i}qz} J_m(\kappa\rho). \tag{1.2}$$

We are using cylindrical polar coordinates (ρ, ϕ, z), with $\rho = (x^2+y^2)^{1/2}$ the distance from the z-axis, and ϕ the azimuthal angle. The azimuthal winding number m is an integer, by continuity of ψ_m; J_m is the regular Bessel function. The wavenumber weight function $f(k,\kappa)$ is in general complex; the constraints on it are that (1.2) should exist, and that physical quantities derived from it (such as the pulse energy and momentum) should be finite.

Although the results given in (1.1) are based entirely on classical electrodynamics, they show that an electromagnetic pulse may be viewed as a superposition of photons with energies $\hbar ck$, z component of momentum $\hbar q = \hbar k_z$, and z component of angular momentum $\hbar m$, where m is the azimuthal quantum number.

The remainder of this section summarizes how electromagnetic pulses may be obtained from solutions of the wave equation. In §2 we give a brief summary of the known closed-form solutions of the wave equation. It is a peculiar fact that most of the known closed-form localized solutions of the wave equation contain both forward- and backward-propagating parts. For example, we shall show localized solutions in §2 propagating in the z-direction, with both $z-ct$ and $z+ct$ in their waveforms. Such pulses are not causal: they cannot originate from a single radiation source or a localized array of sources. However, a general causal wave function, whose construction guarantees the absence of $z+ct$ terms, is readily written down, and some closed-form examples are known. These pulses may be regarded as superpositions of generalized Bessel beams. In §3, a closed-form solution of the wave equation, localized and causal, is found. It is constructed from a superposition of monochromatic beams of a simple type. Electromagnetic TE and TM pulses are discussed in §4, and their energies, momenta and angular momenta evaluated as integrals over their wavenumber weight function in §§5 and 6. Section 7 gives the energy and momentum densities for the two electromagnetic pulses derived from the real and imaginary parts of the wave function found in §3. The energy and momentum densities are different, but the total energy and total momentum are the same. The Discussion in §8 relates the pulse results to those known for electromagnetic beams.

As is well known, electric and magnetic fields can be expressed in terms of the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ and scalar potential $V(\boldsymbol{r},t)$ via

$$\boldsymbol{E} = -\nabla \mathrm{V} - \partial_{ct}\boldsymbol{A}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{1.3}$$

With these substitutions, the source-free Maxwell equations $\nabla.\boldsymbol{B} = 0$, $\nabla\times\boldsymbol{E} + \partial_{ct}\boldsymbol{B} = 0$ are satisfied automatically. If further $\boldsymbol{A}$ and V satisfy the Lorenz condition $\nabla\cdot\boldsymbol{A} + \partial_{ct}V = 0$, substitution of (1.3) into Maxwell's free-space equations (of which the curl equations couple $\boldsymbol{E}$ and $\boldsymbol{B}$) decouples the vector and the scalar potentials, which satisfy the wave equation:

$$\nabla^2\boldsymbol{A} - \partial_{ct}^2\boldsymbol{A} = 0 \quad \text{and} \quad \nabla^2 V - \partial_{ct}^2 V = 0\,. \tag{1.4}$$

Let $\psi_{\mathrm{i}}(\boldsymbol{r}, t)$ be solutions of the wave equation

$$\nabla^2\psi - \partial_{ct}^2\psi = 0. \tag{1.5}$$

Form the four-potential $[\boldsymbol{A}, V] = [\psi_1, \psi_2, \psi_3, \psi_4]$, and the electric and magnetic fields derived from it. Then, provided $\partial_x\psi_1 + \partial_y\psi_2 + \partial_z\psi_3 + \partial_{ct}\psi_4 = 0$, the electric and magnetic fields derived from the four-potential will satisfy the Maxwell equations. (We shall use square brackets for Cartesian coordinates in 3 or 3 + 1 dimensions, round brackets for polar coordinates.) In fact *two* solutions of the wave equation are sufficient to represent an arbitrary electromagnetic field in empty space. This theorem is due to Whittaker [2]; see also [3], §16.2.

Monochromatic scalar beams (and electromagnetic beams constructed from them) have time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$, and hence satisfy the Helmholtz equation

$$\nabla^2\Psi(\boldsymbol{r}, k) + k^2\Psi(\boldsymbol{r}, k) = 0, \quad k = \frac{\omega}{c}. \tag{1.6}$$

We shall see in the next section that pulses may usefully be viewed as superpositions of beams.

2. Solutions of the wave equation in cylindrical coordinates

We briefly summarize the known closed-form solutions of the wave equation, which in cylindrical coordinates reads

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\phi^2 + \partial_z^2 - \partial_{ct}^2\right)\psi(\rho, \phi, z, t) = 0. \tag{2.1}$$

The separable solutions of (2.1) are

$$J_m(\kappa\rho)\mathrm{e}^{\mathrm{i}m\phi}\mathrm{e}^{\mathrm{i}qz}\mathrm{e}^{-\mathrm{i}kct} \quad \text{if} \quad \kappa^2 + q^2 = k^2. \tag{2.2}$$

Superposition of such solutions gives the general causal pulse

$$\psi_m(\rho, \phi, z, t) = \mathrm{e}^{\mathrm{i}m\phi}\int_0^\infty \mathrm{d}k\,\mathrm{e}^{-\mathrm{i}kct}\int_0^k \mathrm{d}\kappa f(k, \kappa)\mathrm{e}^{\mathrm{i}qz}J_m(\kappa\rho), \quad q = \sqrt{k^2 - \kappa^2}. \tag{2.3}$$

The function $f(k, \kappa)$, in general complex, is subject only to the existence of (2.3) and associated integrals, namely those which give the total energy, momentum and angular momentum of a pulse constructed from ψ_m. The form of (2.3) guarantees the absence of $z + ct$ terms: the integrand contains the factor $\mathrm{e}^{\mathrm{i}(qz-kct)}$, with $k \geq q \geq 0$. In §3, we shall give a particular closed-form evaluation of (2.3); earlier closed-form expressions were obtained by Sheppard & Saari [4] and by Zamboni-Rached [5].

The Bessel-based pulses in (2.3) are related to *generalized Bessel beams* [6–8]. For integral m, we define a scalar monochromatic beam of angular frequency $\omega = kc$ by

$$\Psi_m(\rho, \phi, z, k) = \mathrm{e}^{\mathrm{i}m\phi}\int_0^k \mathrm{d}\kappa f(k, \kappa)\mathrm{e}^{\mathrm{i}qz}J_m(\kappa\rho), \quad q = \sqrt{k^2 - \kappa^2}. \tag{2.4}$$

Such beam wave functions automatically satisfy the Helmholtz equation $(\nabla^2 + k^2)\Psi(\mathbf{r}) = 0$. The pulse (2.3) is seen to be a superposition of generalized Bessel beams:

$$\psi_m(\rho, \phi, z, t) = \int_0^\infty \mathrm{d}k\,\mathrm{e}^{-\mathrm{i}kct}\Psi_m(k, \rho, z). \tag{2.5}$$

It is interesting that the intuitive idea of a beam being a superposition of pulses is thus reversed. The physical meaning of (2.5) is remarkable, though typical of Fourier analysis: a continuum of monochromatic beams with phase factor $\mathrm{e}^{-\mathrm{i}kct}$, each longitudinally infinite, adds up to a localized pulse.

Bateman [9] obtained a *general solution of the wave equation in integral form*. For solutions with axial symmetry (independent of the azimuthal angle ϕ) the non-singular part of the Bateman solution is, with $F(u, v)$ any twice-differentiable function,

$$\psi(\rho, z, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, F(z + \mathrm{i}\rho\cos\theta, ct + \rho\sin\theta)\,. \tag{2.6}$$

A proof (different from Bateman's) that (2.6) satisfies the wave equation is given in [10], p. 491.

On the propagation axis $\rho = 0$, the pulse wave function (2.6) becomes

$$\psi(0, z, t) = F(z, t). \tag{2.7}$$

For example, if the on-axis wave function takes the form $\mathrm{e}^{\mathrm{i}k(z-ct)}$, Bateman's integral becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}k(z-ct+\mathrm{i}\rho\mathrm{e}^{\mathrm{i}\theta})} = \frac{\mathrm{e}^{\mathrm{i}k(z-ct)}}{2\pi\mathrm{i}}\oint \mathrm{d}u \frac{\mathrm{e}^{-k\rho u}}{u} = \mathrm{e}^{\mathrm{i}k(z-ct)} \tag{2.8}$$

($u = \mathrm{e}^{\mathrm{i}\theta}$ and the second integral is around the unit circle, with a simple pole at $u = 0$). Thus the plane-wave form on the axis determines the whole pulse to be a plane wave, unidirectional but not localized.

To obtain a pulse localized in space–time, we need to start with a beam which is transversely localized. A closed-form example of a transversely localized beam is provided by a simple wave function, though with problems [11]

$$\Psi(\boldsymbol{r}, k) = \mathrm{e}^{-ka'} R^{-1} \sin kR, \quad R^2 = \rho^2 + (z - \mathrm{i}b')^2. \tag{2.9}$$

Equation (2.5) then gives us

$$\psi(\boldsymbol{r}, t) = \frac{1}{2\mathrm{i}R}\int_0^{\infty} \mathrm{d}k \mathrm{e}^{-ka'} \mathrm{e}^{-\mathrm{i}kct}(\mathrm{e}^{\mathrm{i}kR} - \mathrm{e}^{-\mathrm{i}kR}) = \frac{1}{R^2 + (a' + \mathrm{i}ct)^2}. \tag{2.10}$$

We recognize this wave function as a variant of the solution $(r^2 - c^2t^2)^{-1}$ of the wave equation. This solution is singular on the light-cone $r^2 = c^2t^2$, but complex displacements in space and time make it non-singular, an idea credited by Trautman [12] to Synge [13]. We set

$$a' = \frac{a + b}{2}, \quad b' = -\frac{a - b}{2}. \tag{2.11}$$

Then (2.10) reduces to [14,15]

$$\psi(\boldsymbol{r}, t) = \frac{1}{\rho^2 + [a - \mathrm{i}(z + ct)][b + \mathrm{i}(z - ct)]} = \frac{1}{r^2 - (ct)^2 + ab + \mathrm{i}[(a - b)z - (a + b)ct]}. \tag{2.12}$$

Ziolkowski [14] discusses cylindrically symmetric solutions of the wave equation of the form

$$\psi_{\mathrm{Z}}(\rho, z, t) = [b + \mathrm{i}(z - ct)]^{-1}\int_0^{\infty} \mathrm{d}k\, F(k)\mathrm{e}^{\mathrm{i}k(z+ct) - k\rho^2/[b + \mathrm{i}(z-ct)]}. \tag{2.13}$$

The choice $F(k) = \mathrm{e}^{-ka}$ reproduces the simple wave function (2.12). As noted in [15,16], the wave function (2.12) is predominantly forward propagating for $a \gg b$.

Hillion [17] notes that the wave equation is solved by functions of the form

$$\psi_{\mathrm{H}}(\rho, z, t) = \frac{f(s)}{b + \mathrm{i}(z - ct)}, \quad s = \frac{\rho^2}{b + \mathrm{i}(z - ct)} - \mathrm{i}(z + ct). \tag{2.14}$$

With $f(s) = 1/(s + a)$ we regain the wave function (2.12). An oscillatory wave function results from the choice $f(s) = \mathrm{e}^{-ks}/(s + a)$ [18]:

$$\psi(\rho, z, t) = \frac{\mathrm{e}^{\mathrm{i}k(z+ct) - (k\rho^2/b + \mathrm{i}(z-ct))}}{\rho^2 + [a - \mathrm{i}(z + ct)][b + \mathrm{i}(z - ct)]}. \tag{2.15}$$

The Hillion form of solutions given in (2.14) can be generalized to give pulses with *azimuthal dependence* [19–21]. One can verify, for example, that the following are wave functions:

$$\frac{g}{b+\mathrm{i}(z-ct)}\psi_{\mathrm{H}} \quad (g=x,\ y,\ z,\ x\pm \mathrm{i}y), \qquad \frac{h}{[b+\mathrm{i}(z-ct)]^2}\psi_{\mathrm{H}} \quad (h=xy,\ x^2-y^2,\ (x\pm \mathrm{i}y)^2). \tag{2.16}$$

We note that all of the closed-form solutions shown above contain both $z-ct$ and $z+ct$.

Bateman's integral solution (2.6) is directly related to the $m=0$ form of (2.4). On the z-axis, we have

$$\psi_0(0,z,t)=\int_0^\infty \mathrm{d}k\mathrm{e}^{-\mathrm{i}kct}\int_0^k \mathrm{d}\kappa f(k,\kappa)\mathrm{e}^{\mathrm{i}qz}=F(z,t), \quad q=\sqrt{k^2-\kappa^2}. \tag{2.17}$$

The full wave function is thus, on substitution into (2.6),

$$\psi_0(\rho,z,t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta\int_0^v \mathrm{d}k\mathrm{e}^{-\mathrm{i}k(ct+\rho\sin\theta)}\int_0^k \mathrm{d}\kappa\, f(k,\kappa)\mathrm{e}^{\mathrm{i}q(z+\mathrm{i}\rho\cos\theta)}. \tag{2.18}$$

By Bessel's integral (Watson [22], §2.21), the integration over θ gives $2\pi J_0(\kappa\rho)$, and we regain the $m=0$ form of (2.3):

$$\psi_0(\rho,\phi,z,t)=\int_0^\infty \mathrm{d}k\mathrm{e}^{-\mathrm{i}kct}\int_0^k \mathrm{d}\kappa f(k,\kappa)\mathrm{e}^{\mathrm{i}qz}J_0(\kappa\rho), \quad q=\sqrt{k^2-\kappa^2}. \tag{2.19}$$

The norm of ψ_m defined in (2.3) is evaluated in appendix A. It is

$$N=\int \mathrm{d}^3r\,|\psi_m|^2=(2\pi)^2\int_0^\infty \mathrm{d}k\,k^{-1}\int_0^k \mathrm{d}\kappa\,\kappa^{-1}\sqrt{k^2-\kappa^2}\,|f(k,\kappa)|^2. \tag{2.20}$$

Note that the norm is independent of time. An interesting aspect of the result (2.20) is that the normalization integral does not depend on the azimuthal index m, assuming that the weight function $f(k,\kappa)$ is independent of m. We see also that the function $f(k,\kappa)$ cannot be chosen arbitrarily. For example, logarithmic divergence would result from a non-zero value of $f(k,0)$, making a finite norm impossible.

3. A particular causal pulse

We saw in §2 that, if $\Psi(\boldsymbol{r},k)$ satisfies the Helmholtz equation, a set of pulses satisfying the wave equation can be written in the form

$$\psi(\boldsymbol{r},t)=\int_0^\infty \mathrm{d}k\,\mathrm{e}^{-\mathrm{i}kct}\Psi(\boldsymbol{r},k). \tag{3.1}$$

As the simplest example, if $\Psi(\boldsymbol{r},k)=f(k)\mathrm{e}^{\mathrm{i}kz}$, we obtain the general plane-wave pulse $\psi(z-ct)$. This is not transversely localized; to obtain a pulse localized in space–time we need to start with a beam which is transversely localized.

The simplest causal beam known to the author is the proto-beam [23]; it is the confluent tight-focus limit of two families of beams, both transversely bounded exact solutions of the Helmholtz equation. The first family was introduced by Carter [24] and its properties were explored by Berry [25] and Nye [26]:

$$\Psi_C(\rho,z,k)=\int_0^k \mathrm{d}q\,q\,\mathrm{e}^{(bq^2/2k)+\mathrm{i}qz}\,J_0\left(\rho\sqrt{k^2-q^2}\right). \tag{3.2}$$

The second was considered in [23,27],

$$\Psi_b(\rho,z,k)=\int_0^k \mathrm{d}q\,q\,\mathrm{e}^{qb+\mathrm{i}qz}\,J_0\left(\rho\sqrt{k^2-q^2}\right). \tag{3.3}$$

The lengths k^{-1} and b determine the extent of the focal region. When kb is large the longitudinal extent is of order b, and the transverse extent is of order $\sqrt{b/k}$. As $b\to 0$, the only length remaining is k^{-1}, and both longitudinal and transverse extents of the focal region are of order k^{-1}. This is the case for the confluent limit of Ψ_C, Ψ_b as b tends to zero:

$$\Psi_0(\rho, z, k) = \int_0^k \mathrm{d}q\, q\, \mathrm{e}^{\mathrm{i}qz} J_0\left(\rho\sqrt{k^2 - q^2}\right). \tag{3.4}$$

Explicit expressions for Ψ_0 are given in [23], in terms of Lommel functions of two variables, and alternatively in terms of a series expansion in spherical Bessel functions and Legendre polynomials, as discussed in appendix B. We shall construct a solution of the wave equation from a superposition of $\Psi_0(\rho, z, k)$ with the weight functions $f(k, \kappa)$ or $h(k, q)$ given by

$$f(k, \kappa) = \frac{a^4}{3} k\mathrm{e}^{-ka}\kappa, \quad h(k, q) = q\kappa^{-1} f(k, \kappa) = \frac{a^4}{3} k\mathrm{e}^{-ka} q. \tag{3.5}$$

It is assumed that $a > 0$. The resulting solution of the wave equation is

$$\psi_0(\rho, z, t) = \frac{a^4}{3} \int_0^\infty \mathrm{d}k\, k\, \mathrm{e}^{-ka - \mathrm{i}kct} \int_0^k \mathrm{d}q\, q\, \mathrm{e}^{\mathrm{i}qz} J_0\left(\rho\sqrt{k^2 - q^2}\right). \tag{3.6}$$

The prefactor in the weight functions (3.5) and hence in (3.6) has been chosen to make the wave function unity at the space–time origin: $\psi_0(0, 0, 0) = 1$. In the $z = 0$ plane, we use the integrals

$$\int_0^k \mathrm{d}\kappa\kappa\, J_0(\kappa\rho) = \frac{k}{\rho} J_1(k\rho) \quad \text{and} \quad \int_0^\infty \mathrm{d}k\, k^2\, \mathrm{e}^{-ka} J_1(k\rho) = \frac{3a\rho}{(a^2 + \rho^2)^{5/2}}. \tag{3.7}$$

Hence the wave function in the focal plane $z = 0$ of the pulse is

$$\psi_0(\rho, 0, t) = \frac{a^4 \tilde{a}}{(\tilde{a}^2 + \rho^2)^{5/2}}, \quad \tilde{a} = a + \mathrm{i}ct. \tag{3.8}$$

To evaluate (3.6) at a general space–time point, we first use Bessel's integral ([22], §2.21) to write

$$J_0\left(\rho\sqrt{k^2 - q^2}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta\, \mathrm{e}^{\rho(q\cos\theta + \mathrm{i}k\sin\theta)}. \tag{3.9}$$

The integrations in (3.6) over $0 \le q \le k$ and then over $0 \le k < \infty$ may be reversed to $q \le k < \infty$ and then $0 \le q < \infty$, since $a > 0$. In the final integration over θ an ambiguity arises in the term $\ln(-a - \rho + \mathrm{i}z)$. We interpret this as $\ln(a + \rho - \mathrm{i}z) - \mathrm{i}\pi$ to have agreement with the focal plane value given in (3.8). The final result is

$$\psi_0(\rho, z, t) = \frac{a^4 \tilde{a}}{3} \frac{3(\tilde{a}^2 + \rho^2)^2 - 6z^2(\tilde{a}^2 + \rho^2) - z^4 + 8\mathrm{i}z(\tilde{a}^2 + \rho^2)^{3/2}}{(\tilde{a}^2 + \rho^2)^{3/2}(\tilde{a}^2 + \rho^2 + z^2)^3}, \quad \tilde{a} = a + \mathrm{i}ct. \tag{3.10}$$

Differentiation verifies that ψ_0 satisfies the wave equation. At $t = 0$, the absolute square of ψ_0 is simple:

$$|\psi_0(\rho, z, 0)|^2 = \frac{a^{10}(9a^2 + 9\rho^2 + z^2)}{9(a^2 + \rho^2)^3 (a^2 + \rho^2 + z^2)^3}. \tag{3.11}$$

We note that the lateral decay of $|\psi_0(\rho, z, 0)|$ is faster than the longitudinal decay: asymptotically these are, respectively, ρ^{-5} and z^{-2}. On the propagation axis $\rho = 0$ the modulus squared is

$$|\psi_0(0, z, t)|^2 = \frac{a^8[9a^2 + (z - 3ct)^2]}{9(a^2 + c^2t^2)^2 [a^2 + (z - ct)^2]^3}. \tag{3.12}$$

Propagation is in the positive z-direction. There is no backward propagation part containing $z + ct$ as we had in (2.12–2.16). The term $z - 3ct$ suggests superluminal propagation, but in fact the maximum modulus is at $z \approx (12/13)ct$ in the focal region, and at $z \approx ct$ far from the focal region. Figure 1 shows the time development of the modulus of ψ_0.

We note in conclusion of this section that solutions with azimuthal dependence may be obtained by differentiation of ψ_0: the wave equation operator $\nabla^2 - \partial_{ct}^2$ commutes with ∂_x and

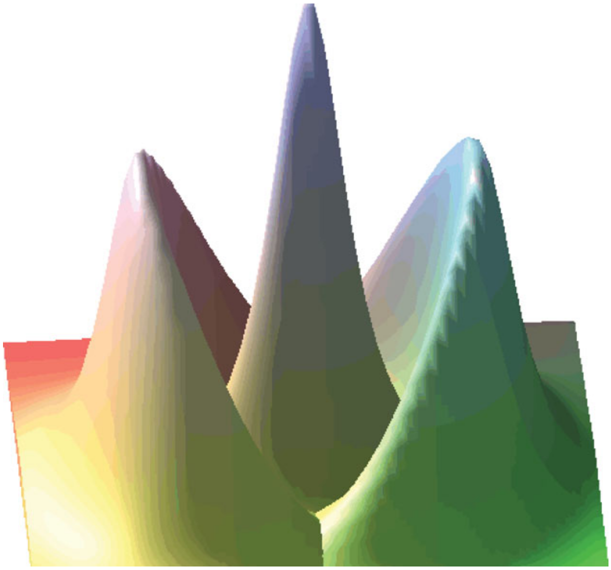

Figure 1. Plots of the modulus $|\psi_0(\rho, z, t)|$ at times $t = 0$, $ct = \pm 3a$. The central $t = 0$ peak has been scaled down by a factor of three so that it would not swamp the neighbouring moduli at $ct = \pm 3a$. Propagation is from left to right. There is symmetry about the space–time origin, which is the centre of the focal region of the pulse. The apparent discontinuity seen at the front centre is due to overprinting of the moduli. (Online version in colour.)

∂_y and thus also with $\partial_x + \mathrm{i}\partial_y = \mathrm{e}^{\mathrm{i}\phi}(\partial_\rho + \mathrm{i}\rho^{-1}\partial_\phi)$. Hence, for example, a solution with $m = 1$ is (since $J_0' = -J_1$)

$$\psi_1 = \mathrm{e}^{\mathrm{i}\phi} \int_0^k \mathrm{d}q\, q\, \mathrm{e}^{\mathrm{i}qz}\, \kappa\, J_1(\rho\kappa), \quad \kappa = \sqrt{k^2 - q^2}. \tag{3.13}$$

An equivalent form is obtained by operating on (3.10) with $-\mathrm{e}^{\mathrm{i}\phi}\partial_\rho$.

4. Electromagnetic transverse electric and transverse magnetic pulses

As we saw in §1, free-space electromagnetic fields can be expressed in terms of the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ and scalar potential $V(\boldsymbol{r}, t)$ which satisfy the wave equation (1.4) and the Lorenz condition $\nabla \cdot \boldsymbol{A} + \partial_{ct} V = 0$.

Electromagnetic pulses can thus be constructed from solutions of (1.4). As a simple example, the choice $V = \text{const.}$, $\boldsymbol{A} = \nabla \times [0, 0, \psi] = [\partial_y, -\partial_x, 0]\psi$ satisfies the Lorenz condition, and gives us the TE pulse with

$$\boldsymbol{E} = -\partial_{ct} A = [-\partial_y\partial_{ct}, \partial_x\partial_{ct}, 0]\psi \quad \text{and} \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} = [\partial_x\partial_z, \partial_y\partial_z, -\partial_x^2 - \partial_y^2]\psi. \tag{4.1}$$

We shall explore in §5 the properties of this pulse when $\psi = \psi_0$, the wave function found in the previous section. But first we look at the general properties of TE pulses. Because of the nature of our solutions, it is convenient to use cylindrical coordinates (ρ, ϕ, z). When ψ is dependent on the azimuthal angle ϕ through the factor $\mathrm{e}^{\mathrm{i}m\phi}$, as in the general causal solution (2.3), we have

$$\partial_x = \cos\phi\,\partial_\rho - \rho^{-1}\sin\phi\,\partial_\phi \to \cos\phi\,\partial_\rho - \mathrm{i}m\rho^{-1}\sin\phi \tag{4.2}$$

and

$$\partial_y = \sin\phi\,\partial_\rho + \rho^{-1}\cos\phi\,\partial_\phi \to \sin\phi\,\partial_\rho + \mathrm{i}m\rho^{-1}\cos\phi. \tag{4.3}$$

Using the fact that ψ satisfies the wave equation, the complex fields simplify to

$$\boldsymbol{E} = -\partial_{ct}\boldsymbol{A} = (-\mathrm{i}m\rho^{-1}\partial_{ct}, \partial_\rho\partial_{ct}, 0)\psi \quad \text{and} \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} = (\partial_\rho\partial_z, \mathrm{i}m\rho^{-1}\partial_z, \partial_z^2 - \partial_{ct}^2)\psi. \tag{4.4}$$

Write the complex wave function as $\psi=\psi_\mathrm{r}+\mathrm{i}\psi_\mathrm{i}$; both the real and the imaginary parts are solutions of the wave equation. Real electric and magnetic fields are obtained by taking the real or the imaginary parts of the complex fields. The real and imaginary parts of (4.4) are, respectively,

$$\boldsymbol{E}=(m\rho^{-1}\partial_{ct}\psi_\mathrm{i},\ \partial_\rho\partial_{ct}\psi_\mathrm{r},\ 0),\quad \boldsymbol{B}=(\partial_\rho\partial_z\psi_\mathrm{r},\ -m\rho^{-1}\partial_z\psi_\mathrm{i},\ \partial_z^2\psi_\mathrm{r}-\partial_{ct}^2\psi_\mathrm{r}) \tag{4.5}$$

and

$$\boldsymbol{E}=(-m\rho^{-1}\partial_{ct}\psi_\mathrm{r},\ \partial_\rho\partial_{ct}\psi_\mathrm{i},\ 0),\quad \boldsymbol{B}=(\partial_\rho\partial_z\psi_\mathrm{i},\ m\rho^{-1}\partial_z\psi_\mathrm{r},\ \partial_z^2\psi_\mathrm{i}-\partial_{ct}^2\psi_\mathrm{i}). \tag{4.6}$$

The energy, momentum and angular momentum densities are, for real fields,

$$u=\frac{1}{8\pi}(E^2+B^2),\quad c\boldsymbol{p}=\frac{1}{4\pi}\boldsymbol{E}\times\boldsymbol{B},\quad \boldsymbol{j}=\boldsymbol{r}\times\boldsymbol{p}. \tag{4.7}$$

A TM pulse is obtained from the TE pulse by the *duality transformation* $\boldsymbol{E}\to\boldsymbol{B},\ \boldsymbol{B}\to-\boldsymbol{E}$. The duality transformation leaves the energy and momentum densities unchanged. Thus for both TE and TM pulses, with the fields given in (4.5), we have

$$8\pi u=(\partial_\rho\partial_z\psi_\mathrm{r})^2+(\partial_\rho\partial_{ct}\psi_\mathrm{r})^2+[\partial_z^2\psi_\mathrm{r}-\partial_{ct}^2\psi_\mathrm{r}]^2+m^2\rho^{-2}[(\partial_z\psi_\mathrm{i})^2+(\partial_{ct}\psi_\mathrm{i})^2], \tag{4.8}$$

$$4\pi cp_\rho=E_\phi B_z-E_zB_\phi=E_\phi B_z=(\partial_\rho\partial_{ct}\psi_\mathrm{r})(\partial_z^2\psi_\mathrm{r}-\partial_{ct}^2\psi_\mathrm{r}), \tag{4.9}$$

$$4\pi cp_\phi=E_zB_\rho-E_\rho B_z=-E_\rho B_z=-m\rho^{-1}(\partial_{ct}\psi_\mathrm{i})(\partial_z^2\psi_\mathrm{r}-\partial_{ct}^2\psi_\mathrm{r}) \tag{4.10}$$

and

$$4\pi cp_z=E_\rho B_\phi-E_\phi B_\rho=-m^2\rho^{-2}(\partial_{ct}\psi_\mathrm{i})(\partial_z\psi_\mathrm{i})-(\partial_\rho\partial_{ct}\psi_\mathrm{r})(\partial_\rho\partial_z\psi_\mathrm{r}). \tag{4.11}$$

The z component of the angular momentum density is therefore proportional to the azimuthal winding number m; for the pulse formed from the real part of the wave function it is

$$j_z=xp_y-yp_x=\rho p_\phi=-\frac{m}{4\pi c}(\partial_{ct}\psi_\mathrm{i})(\partial_z^2\psi_\mathrm{r}-\partial_{ct}^2\psi_\mathrm{r}). \tag{4.12}$$

The energy and momentum densities obtained by choosing the fields (4.6) have the same form, with ψ_r and ψ_i interchanged, and a change of sign in p_ϕ.

However, we shall see that the total energy, momentum and angular momentum are the same whether the real or the imaginary part of ψ is used. The total energies and momenta are obtained by integrating over all space at fixed time:

$$U=\int \mathrm{d}^3r\,u(\boldsymbol{r},t),\quad \boldsymbol{P}=\int \mathrm{d}^3r\,\boldsymbol{p}(\boldsymbol{r},t),\quad \boldsymbol{J}=\int \mathrm{d}^3r\,\boldsymbol{r}\times\boldsymbol{p}(\boldsymbol{r},t). \tag{4.13}$$

We wish to evaluate these spatial integrals in terms of the weight function $f(k,\kappa)$. The calculations are simpler for the case where ψ is independent of the azimuthal angle ϕ, which we look at next.

5. Energy and momentum for the $m=0$ pulse

Let us look at the first term in the energy density of a TE or TM pulse when the real part of the complex fields is used, $(\partial_\rho\partial_z\psi_\mathrm{r})^2$, in which we write $\psi_\mathrm{r}=1/2(\psi+\psi^*)$,

$$(\partial_\rho\partial_z\psi_\mathrm{r})^2=\frac{1}{4}[(\partial_\rho\partial_z\psi)^2+2(\partial_\rho\partial_z\psi)(\partial_\rho\partial_z\psi^*)+(\partial_\rho\partial_z\psi^*)^2]. \tag{5.1}$$

In the integration over all space, the first term gives

$$\int \mathrm{d}^3r\,(\partial_\rho\partial_z\psi)^2=2\pi\int_0^\infty \mathrm{d}\rho\ \rho\int_{-\infty}^\infty \mathrm{d}z$$
$$\int_0^\infty \mathrm{d}k\,\mathrm{e}^{-\mathrm{i}kct}\int_0^k \mathrm{d}\kappa\,f(k,\kappa)\mathrm{i}q\mathrm{e}^{\mathrm{i}qz}\kappa J_1(\kappa\rho)\int_0^\infty \mathrm{d}k'\mathrm{e}^{-\mathrm{i}k'ct}\int_0^{k'}\mathrm{d}\kappa'\,f(k',\kappa')\mathrm{i}q'\mathrm{e}^{\mathrm{i}q'z}\kappa' J_1(\kappa'\rho). \tag{5.2}$$

We refer to appendix A, where more detail is provided in the evaluation of the norm of the pulse. From (A 4), the integration over z gives rise to delta functions, which in the case of the squares $(\partial_\rho\partial_z\psi)^2$ and $(\partial_\rho\partial_z\psi^*)^2$ are both $\delta(q+q')$. As q and q' are non-negative, these terms integrate to zero. Only the term $(1/2)(\partial_\rho\partial_z\psi)(\partial_\rho\partial_z\psi^*)$ remains. The integration over ϕ gives the factor 2π. Hence we have

$$\begin{aligned}\int \mathrm{d}^3r\,(\partial_\rho\partial_z\psi_\mathrm{r})^2 &= \frac{1}{2}\int \mathrm{d}^3r(\partial_\rho\partial_z\psi^*)(\partial_\rho\partial_z\psi) = \pi\int_0^\infty \mathrm{d}\rho\;\rho\int_{-\infty}^\infty \mathrm{d}z(\partial_\rho\partial_z\psi^*)(\partial_\rho\partial_z\psi)\\ &= \pi\int_0^\infty \mathrm{d}\rho\;\rho\int_{-\infty}^\infty \mathrm{d}z\int_0^\infty \mathrm{d}k\mathrm{e}^{\mathrm{i}kct}\int_0^k \mathrm{d}\kappa\, f(k,\kappa)^* q\mathrm{e}^{-\mathrm{i}qz}\kappa J_1(\kappa\rho)\\ &\quad\times\int_0^\infty \mathrm{d}k'\mathrm{e}^{-\mathrm{i}k'ct}\int_0^{k'} \mathrm{d}\kappa'\, f(k',\kappa')q'\mathrm{e}^{\mathrm{i}q'z}\kappa' J_1(\kappa'\rho).\end{aligned} \tag{5.3}$$

Performing the integration over ρ gives us, with the use of (A 2),

$$\pi\int_{-\infty}^\infty \mathrm{d}z\int_0^\infty \mathrm{d}k\,\mathrm{e}^{\mathrm{i}kct}\int_0^k \mathrm{d}\kappa\, f(k,\kappa)^* q\mathrm{e}^{-\mathrm{i}qz}\kappa\int_0^\infty \mathrm{d}k'\mathrm{e}^{-\mathrm{i}k'ct} f(k',\kappa)q'\mathrm{e}^{\mathrm{i}q'z}. \tag{5.4}$$

Finally, the integration over z with the use of (A 4) gives

$$\int \mathrm{d}^3r\,(\partial_\rho\partial_z\psi_\mathrm{r})^2 = 2\pi^2\int_0^\infty \mathrm{d}k\int_0^k \mathrm{d}\kappa\,|f(k,\kappa)|^2 k^{-1}\kappa q^3. \tag{5.5}$$

The same expression results if we choose to use the imaginary part $\psi_\mathrm{i} = (1/2i)(\psi - \psi^*)$.

Next we shall evaluate

$$\int \mathrm{d}^3r\,(\partial_\rho\partial_{ct}\psi_\mathrm{r})^2 = \int \mathrm{d}^3r\,(\partial_\rho\partial_{ct}\psi_\mathrm{i})^2 = \frac{1}{2}\int \mathrm{d}^3r(\partial_\rho\partial_{ct}\psi^*)(\partial_\rho\partial_{ct}\psi). \tag{5.6}$$

The differentiations with respect to ρ bring down the factors κ, κ', and the differentiations with respect to ct the factors $\mathrm{i}k$, $-\mathrm{i}k'$. The integration over ρ gives $\kappa^{-1}\delta(\kappa-\kappa')$ from (A2), integration over z gives $2\pi qk^{-1}\delta(k-k')$ from (A 4). Hence

$$\int \mathrm{d}^3r\,(\partial_\rho\partial_{ct}\psi_\mathrm{r})^2 = 2\pi^2\int_0^\infty \mathrm{d}k\int_0^k \mathrm{d}\kappa\,|f(k,\kappa)|^2 k\kappa q. \tag{5.7}$$

Likewise,

$$\int \mathrm{d}^3r[\partial_z^2\psi_\mathrm{r} - \partial_{ct}^2\psi_\mathrm{r}]^2 = \frac{1}{2}\int \mathrm{d}^3r(\partial_z^2\psi^* - \partial_{ct}^2\psi^*)(\partial_z^2\psi - \partial_{ct}^2\psi). \tag{5.8}$$

The differentiations give the factors $k^2 - q^2 = \kappa^2$ and κ'^2, so

$$\int \mathrm{d}^3r[\partial_z^2\psi_\mathrm{r} - \partial_{ct}^2\psi_\mathrm{r}]^2 = 2\pi^2\int_0^\infty \mathrm{d}k\int_0^k \mathrm{d}\kappa\,|f(k,\kappa)|^2 k^{-1}\kappa^3 q. \tag{5.9}$$

The *total energy* for the $m=0$ TE and TM pulses based on ψ_r or on ψ_i is thus

$$U = \frac{\pi}{2}\int_0^\infty \mathrm{d}k\int_0^k \mathrm{d}\kappa\,|f(k,\kappa)|^2\, k\kappa q \quad (m=0). \tag{5.10}$$

In the *total momentum* calculation, we need to integrate over $-(\partial_\rho\partial_{ct}\psi_\mathrm{r})(\partial_\rho\partial_z\psi_\mathrm{r})$. By the arguments presented above,

$$\int \mathrm{d}^3r\,(\partial_\rho\partial_{ct}\psi_\mathrm{r})(\partial_\rho\partial_z\psi_\mathrm{r}) = \frac{1}{2}\int \mathrm{d}^3r(\partial_\rho\partial_{ct}\psi^*)(\partial_\rho\partial_z\psi). \tag{5.11}$$

The differentiations give the factors κ, κ', $\mathrm{i}k$, $\mathrm{i}q'$, so

$$cP_z = \frac{\pi}{2}\int_0^\infty \mathrm{d}k\int_0^k \mathrm{d}\kappa\,|f(k,\kappa)|^2\kappa q^2 \quad (m=0). \tag{5.12}$$

The *angular momentum* of the $m=0$ pulse is zero, since the x and y components of $\boldsymbol{j} = \boldsymbol{r}\times\boldsymbol{p}$ contain $\cos\phi$ or $\sin\phi$ factors and integration over ϕ gives zero, and the z component is $j_z = \rho p_\phi$, which is zero.

6. Energy, momentum and angular momentum for $m \neq 0$ pulses

We now consider TE or TM pulses when the azimuthal dependence of ψ is $\mathrm{e}^{\mathrm{i}m\phi}$. In the integration over ϕ the terms $(\partial_\rho \partial_z \psi)^2$ and $(\partial_\rho \partial_z \psi^*)^2$ will give zero, because of the $\mathrm{e}^{\pm 2\mathrm{i}m\phi}$ factors they carry. In the energy density given in (4.8), we shall consider together the terms $(\partial_\rho \partial_z \psi_\mathrm{r})^2 + m^2\rho^{-2}(\partial_z \psi_\mathrm{i})^2$ and $(\partial_\rho \partial_{ct} \psi_\mathrm{r})^2 + m^2\rho^{-2}(\partial_{ct} \psi_\mathrm{i})^2$. The terms surviving the spatial integrals are indicated by arrows:

$$(\partial_\rho \partial_z \psi_\mathrm{r})^2 + m^2\rho^{-2}(\partial_z \psi_\mathrm{i})^2 \to \frac{1}{2}(\partial_\rho \partial_z \psi^*)(\partial_\rho \partial_z \psi) + \frac{m^2}{2\rho^2}(\partial_z \psi^*)(\partial_z \psi) \tag{6.1}$$

and

$$(\partial_\rho \partial_{ct} \psi_\mathrm{r})^2 + m^2\rho^{-2}(\partial_{ct} \psi_\mathrm{i})^2 \to \frac{1}{2}(\partial_\rho \partial_{ct} \psi^*)(\partial_\rho \partial_{ct} \psi) + \frac{m^2}{2\rho^2}(\partial_{ct} \psi^*)(\partial_{ct} \psi). \tag{6.2}$$

The differentiations with respect to z in (6.1) give the factor $(-\mathrm{i}q)(\mathrm{i}q') = qq'$, and the expression in the integrand is proportional to

$$\partial_\rho J_m(\kappa\rho)\,\partial_\rho J_m(\kappa'\rho) + m^2\rho^{-2} J_m(\kappa\rho) J_m(\kappa'\rho). \tag{6.3}$$

We use the recurrence formulae ([22], §2.13)

$$\left.\begin{aligned} 2J'_m(\zeta) &= J_{m-1}(\zeta) - J_{m+1}(\zeta) \\ \text{and} \qquad \frac{2m}{\zeta} J_m(\zeta) &= J_{m-1}(\zeta) + J_{m+1}(\zeta). \end{aligned}\right\} \tag{6.4}$$

These reduce (6.3) to a form amenable to the use of the Hankel inversion formula (A 2), namely

$$\frac{\kappa\kappa'}{2}\left\{J_{m-1}(\kappa\rho)J_{m-1}(\kappa'\rho) + J_{m+1}(\kappa\rho)J_{m+1}(\kappa'\rho)\right\}. \tag{6.5}$$

Integration of (6.5) over ρ by means of (A 2) thus gives the factor κ. Hence

$$\int \mathrm{d}^3 r \left\{(\partial_\rho \partial_z \psi_\mathrm{r})^2 + m^2\rho^{-2}(\partial_z \psi_\mathrm{i})^2\right\} = 2\pi^2 \int_0^\infty \mathrm{d}k \int_0^k \mathrm{d}\kappa\, |f(k,\kappa)|^2 k^{-1} \kappa q^3. \tag{6.6}$$

Note that this is the same value as was obtained in the $m = 0$ case. Likewise,

$$\int \mathrm{d}^3 r \left\{(\partial_\rho \partial_{ct} \psi_\mathrm{r})^2 + m^2\rho^{-2}(\partial_{ct} \psi_\mathrm{i})^2\right\} = 2\pi^2 \int_0^\infty \mathrm{d}k \int_0^k \mathrm{d}\kappa\, |f(k,\kappa)|^2 k\kappa q. \tag{6.7}$$

Again this is the same value as was obtained in the $m = 0$ case. Hence the total energy is given by the expression obtained in the previous section, in equation (5.10). In the calculation on the net total momentum by integration over the momentum density p_z given in (4.11), the same mathematical reduction gives us expression (5.12) for P_z, again unchanged.

It remains to calculate the total angular momentum. The component of interest is J_z, since it is intrinsic to the pulse, unchanged by a Lorentz boost along the propagation direction [28].

From (4.10) and (4.12), we see that the angular momentum density is given by

$$j_z = \rho p_\phi = -\frac{m}{4\pi c}(\partial_{ct}\psi_\mathrm{i})(\partial_z^2 \psi_\mathrm{r} - \partial_{ct}^2 \psi_\mathrm{r}). \tag{6.8}$$

We replace ψ_r, ψ_i by $(1/2)(\psi + \psi^*)$, $(1/2i)(\psi - \psi^*)$ as we have done before. The azimuthal dependence of ψ is $\mathrm{e}^{\mathrm{i}m\phi}$, and the integration over ϕ of the terms carrying $\mathrm{e}^{\pm 2\mathrm{i}m\phi}$ factors gives zero. Hence

$$j_z = \rho p_\phi = \frac{m}{8\pi c}\mathrm{Im}\left\{(\partial_{ct}\psi^*)(\partial_z^2 \psi - \partial_{ct}^2 \psi)\right\} + \text{terms integrating to zero.} \tag{6.9}$$

The differentiations give the factor $\mathrm{i}k(k'^2 - q'^2) = \mathrm{i}k\kappa'^2$. The integration over all space then results in

$$cJ_z = \frac{m\pi}{2}\int_0^\infty \mathrm{d}k \int_0^k \mathrm{d}\kappa\, |f(k,\kappa)|^2 \kappa q. \tag{6.10}$$

The angular momentum is the same whether the real or the imaginary part of the complex wave function is used, since both result in the integrand (6.9). Let us summarize the results obtained

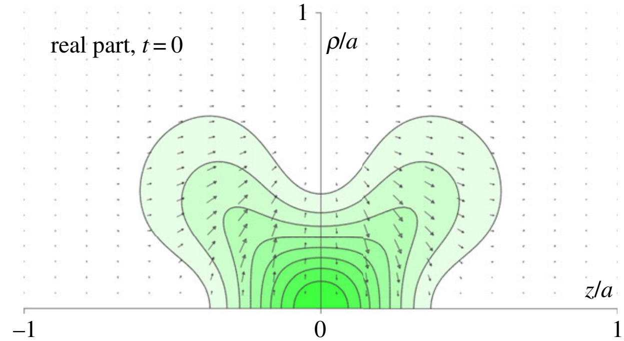

Figure 2. Energy and momentum densities at $t = 0$ corresponding to the real part ψ_r of the wave function (3.10). The energy density is shown by shading and contours, the momentum density by arrows. The momentum density is zero on the axis. The three-dimensional picture is obtained by rotating the figure about the direction of propagation (horizontal axis). (Online version in colour.)

for the TE or TM pulses:

$$\begin{bmatrix} U \\ cP_z \\ cJ_z \end{bmatrix} = \frac{\pi}{2}\int_0^\infty \mathrm{d}k \int_0^k \mathrm{d}\kappa\, |f(k,\kappa)|^2 \kappa q \begin{bmatrix} k \\ q \\ m \end{bmatrix}. \tag{6.11}$$

For comparison, the norm of the wave function (calculated in appendix A) is

$$N \equiv \int \mathrm{d}^3 r\, |\psi|^2 = (2\pi)^2 \int_0^\infty \mathrm{d}k \int_0^k \mathrm{d}\kappa\, |f(k,\kappa)|^2 k^{-1}\kappa^{-1} q. \tag{6.12}$$

Again there is no dependence on the azimuthal index m when $f(k,\kappa)$ is independent of m. Incidentally, we have proved that U, P_z, J_z and N are all constant in time, and verified that $U > cP_z$.

7. TE pulses derived from $\psi_0(\rho, z, t)$

This section gives the energy and momentum for the TE pulses derived from the wave function ψ_0 given in (3.10). There are two pulses, one based on the real and one on the imaginary part of ψ_0. The energy and momentum densities for the real part of a general ψ were given in equations (4.8)–(4.11). Those for the imaginary part are the same expressions with ψ_r replaced by ψ_i (when $m = 0$, which is the case being considered). The total energies and momenta are the same, but the densities are different. The angular momentum is zero.

The weight function $f(k,\kappa)$ is defined in (3.5), and normalizes the wave function ψ_0 to unity at the space–time origin. For this weight function, $N = \pi^2 a^3/9$ is an effective volume of the pulse. Here we are dealing with physical electromagnetic pulses derived from ψ_0. In the pulse based on the real part of ψ_0, we shall normalize according to the fields at the space–time origin. All components given in (4.5) are zero except B_z, which takes the value $B_0 = 10/a^2$. The corresponding energy density at the space–time origin is $u_0 = B_0^2/8\pi$, and the quantity $u_0 N = 25\pi/18a$ provides an energy scale. Keeping the weight function as defined in (3.5), namely $f(k,\kappa) = (a^4/3)k\mathrm{e}^{-ka}\kappa$, we find (by direct spatial integration or from (6.11))

$$U = \frac{7\pi}{12a}, \quad cP_z = \frac{35\pi}{96a}, \quad \frac{cP_z}{U} = \frac{5}{8}. \tag{7.1}$$

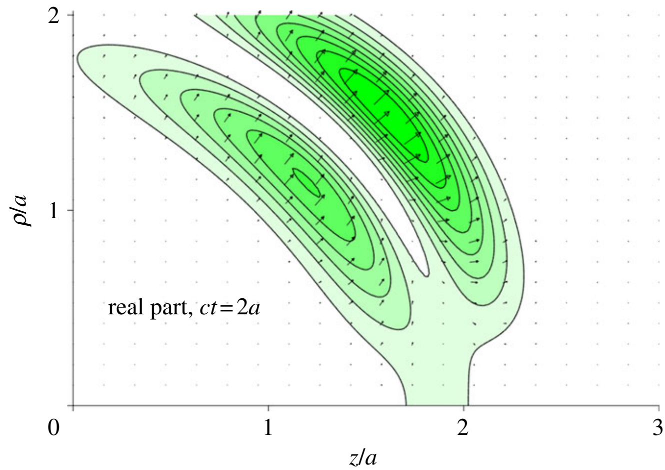

Figure 3. Energy and momentum densities at $ct = 2a$ corresponding to the real part ψ_r of the wave function (3.10). Notation as in figure 2. The location of the pair of density maxima of the annular pulse is discussed in the text. The pulse converges onto or diverges from the axis of propagation, asymptotically on a cone of half-angle equal to 45°. (Online version in colour.)

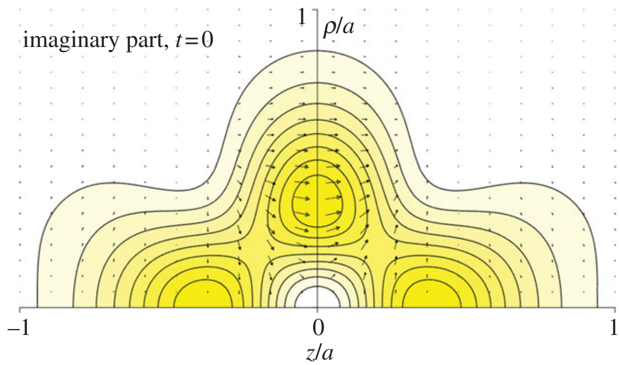

Figure 4. Energy and momentum densities at $t = 0$ corresponding to the imaginary part ψ_i of the wave function (3.8). As in figure 2, the energy density is shown by shading and contours, the momentum density by arrows. Both are zero at the origin. (Online version in colour.)

In terms of the energy unit u_0N,

$$U = \frac{21}{50}u_0N, \quad cP_z = \frac{21}{80}u_0N. \tag{7.2}$$

These values hold for TE pulses formed from either the real or the imaginary parts of ψ_0.

We shall look at the energy and momentum densities separately, starting with the pulse based on the *real part*. Its energy density is maximal at the space–time origin (figure 2). The pulse splits into two annular energy and momentum density maxima travelling together as time progresses, as shown in figure 3. Far from the origin (which is the centre of the focal region) the minimum between the peaks is at $r = ct$, and the energy density peaks are located at $r = ct \pm a\sqrt{7 - 4\sqrt{3}} \approx ct \pm 0.268a$. Their separation is thus asymptotically about $0.536a$.

Next we look at the TE pulse derived from the *imaginary part* of (3.10). Figure 4 shows that the energy and momentum densities derived from ψ_i are zero at the space–time origin. At $t = 0$, the pulse is centred on the origin, with two energy density maxima on the axis at $z = \pm a/\sqrt{7} \approx \pm 0.378a$, and a ring of maximal energy density at $\rho \approx 0.353a$. As time increases (figure 5 shows

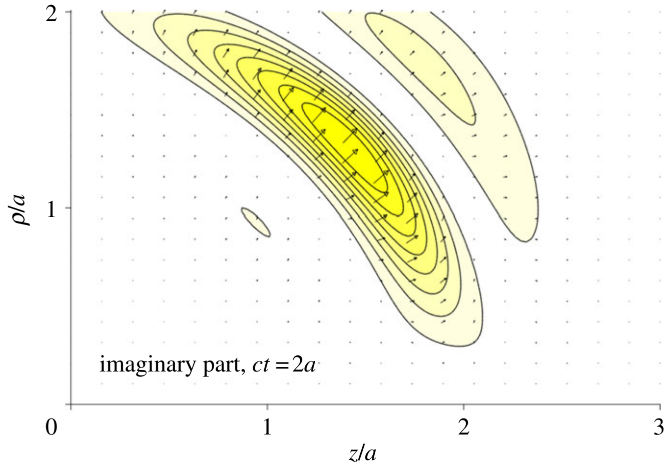

Figure 5. Energy and momentum densities at $ct = 2a$ corresponding to the imaginary part ψ_i of the wave function (3.10). Away from the focal region a single-peak annular structure is established, with maximum energy density at $r = ct$, diverging at 45° to the axis of propagation. (Online version in colour.)

the densities at $ct = 2a$) the pulse becomes predominantly annular, with a single energy and momentum density peak at $r = ct$. The annular pulse diverges from the axis of propagation, asymptotically on a cone of half-angle equal to 45°.

The real part and the imaginary part of ψ both give a TE pulse which is hollow in momentum: the momentum density is zero on the axis of propagation. Both pulses have a focal region of extent a, more compact in the case of the real part. Both converge onto or diverge from the axis of propagation at 45° for $r \gg a$. The dual TM pulses are the same in their energy and momentum densities, and thus also in their total energy and momentum.

8. Discussion

The results of §7 for TE and TM pulses were summarized in equations (6.11), which we repeat here:

$$\begin{bmatrix} U \\ cP_z \\ cJ_z \end{bmatrix} = \frac{\pi}{2} \int_0^\infty \mathrm{d}k \int_0^k \mathrm{d}\kappa\, |f(k,\kappa)|^2 \kappa q \begin{bmatrix} k \\ q \\ m \end{bmatrix}. \tag{8.1}$$

These equations have a simple interpretation in terms of the light quantum: the TE or TM pulses can be viewed as a superposition of photons, each with energy $\hbar ck$, z component of momentum $\hbar q$ and z component of angular momentum $\hbar m$.

It is interesting to compare (8.1) with the expressions for the energy, momentum and angular momentum *per unit length* of electromagnetic beams derived from the general beam wave function (2.3), namely

$$\Psi_m(\rho,\phi,z,k) = \mathrm{e}^{\mathrm{i}m\phi} \int_0^k \mathrm{d}\kappa f(k,\kappa) \mathrm{e}^{\mathrm{i}qz} J_m(\kappa\rho), \quad q = \sqrt{k^2 - \kappa^2}. \tag{8.2}$$

We denote by U' the energy per unit length, and likewise for P'_z, J'_z. The results found in [8] for TE or TM beams are

$$\begin{bmatrix} U' \\ cP'_z \\ cJ'_z \end{bmatrix} = \frac{1}{4k} \int_0^k \mathrm{d}\kappa\, |f(k,\kappa)|^2 \begin{bmatrix} k \\ q \\ m \end{bmatrix}. \tag{8.3}$$

Again, electromagnetic TE and TM beams can be viewed as superpositions of photons, each with energy $\hbar ck$, z component of momentum $\hbar q$ and z component of angular momentum $\hbar m$. However, this very direct correspondence of classical electromagnetic pulses and beams with superpositions of photons holds only in the TE and TM cases. More complicated relations hold for other beams and pulses, as can be seen from the examples given in table 1 of [8].

The (idealized) beam is a spatially static entity, while the pulse is inherently dynamic and changes position and shape as time evolves. In both cases, there are conserved quantities: the pulse energy, momentum and angular momentum are independent of time, and the beam energy, momentum and angular momentum per unit length are independent of position along the beam. (The last statement is for generalized Bessel beams; for discussion of the set of electromagnetic beam invariants, see [8,29].) That, for electromagnetic pulses, $U^2 - c^2P^2$ is a Lorentz invariant and $[c\boldsymbol{P}, U]$ is a four-vector was proved by von Laue in 1911 [30]. See also Griffiths [31], and, for full detail, Møller [32], §63. The scalar product of the four-vector $[c\boldsymbol{P}, U]$ with itself is a scalar, an invariant. Just as $\sqrt{U^2 - c^2P^2} = Mc^2$ is the invariant rest energy for particles, so $\sqrt{U^2 - c^2P^2} = U_0$ is an invariant energy for any given electromagnetic pulse. A Lorentz boost at speed c^2P_z/U along the z-axis will take us to the zero-momentum frame of the pulse (not the 'rest' frame, waves are never at rest) as described in [33].

In §3, we gave a closed-form, strictly causal complex wave function, with only forward propagation. The wave function is characterized by one length, a. The energy and momentum of electromagnetic TE and TM pulses based on real or imaginary parts of this wave function were evaluated (the angular momentum is zero), both by direct integration of the energy and momentum densities, and from the integrals over the modulus squared of the weight function $f(k,\kappa)$. Graphs and analytics of the pair of pulses derived from the real and the imaginary parts of the velocity potential show strong convergence/divergence, with a tight focal region of size a.

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Appendix A. The norm of a wave function

We consider two equivalent expressions for the norm of ψ_m, as defined and evaluated in the following:

$$\begin{aligned} N &= \int \mathrm{d}^3r\, |\psi_m|^2 = \int_0^\infty \mathrm{d}\rho\ \rho \int_0^{2\pi} \mathrm{d}\phi \int_{-\infty}^\infty \mathrm{d}z\, |\psi_m|^2 \\ &= 2\pi \int_0^\infty \mathrm{d}\rho\ \rho \int_{-\infty}^\infty \mathrm{d}z \int_0^\infty \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kct} \int_0^k \mathrm{d}\kappa\, f(k,\kappa)^* \mathrm{e}^{-\mathrm{i}qz} J_m(\kappa\rho) \int_0^\infty \mathrm{d}k' \mathrm{e}^{-\mathrm{i}k'ct} \int_0^{k'} \mathrm{d}\kappa'\, f(k',\kappa') \mathrm{e}^{\mathrm{i}q'z} J_m(\kappa'\rho). \end{aligned} \tag{A 1}$$

Hankel's inversion formula ([22], §14.4, [8], appendix A) may be written as

$$\int_0^\infty \mathrm{d}\rho\ \rho\, J_m(\kappa\rho)\, J_m(\kappa'\rho) = \kappa^{-1}\delta(\kappa - \kappa') \quad (\kappa, \kappa' > 0). \tag{A 2}$$

Hence the integration over ρ selects $\kappa' = \kappa$, and we are left with

$$2\pi \int_{-\infty}^\infty \mathrm{d}z \int_0^\infty \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kct} \int_0^\infty \mathrm{d}k' \mathrm{e}^{-\mathrm{i}k'ct} \int_0^k \mathrm{d}\kappa\, \kappa^{-1} f(k,\kappa)^* f(k',\kappa)\, \mathrm{e}^{\mathrm{i}z\left(\sqrt{k'^2-\kappa^2} - \sqrt{k^2-\kappa^2}\right)}. \tag{A 3}$$

Next we perform the z-integration, and use the Fourier inversion formula

$$\int_{-\infty}^\infty \mathrm{d}z\, \mathrm{e}^{\mathrm{i}z(q'-q)} = 2\pi\ \delta(q'-q) = 2\pi\ \delta\left(\sqrt{k'^2-\kappa^2} - \sqrt{k^2-\kappa^2}\right) = 2\pi\, \frac{\sqrt{k^2-\kappa^2}}{k} \delta(k'-k). \tag{A 4}$$

The last equality follows from the relation (which assumes $G(k)$ to be monotonic in k, so $G(k') = G(k)$ when $k' = k$ and nowhere else)

$$\delta(G(k') - G(k)) = \left|\frac{\mathrm{d}k}{\mathrm{d}G}\right| \delta(k' - k). \tag{A 5}$$

Thus the final result is just a double integral over the absolute square of the function $f(k, \kappa)$:

$$N = \int \mathrm{d}^3r\, |\psi_m|^2 = (2\pi)^2 \int_0^\infty \mathrm{d}k\, k^{-1} \int_0^k \mathrm{d}\kappa\, \kappa^{-1} \sqrt{k^2 - \kappa^2}\, |f(k,\kappa)|^2. \tag{A 6}$$

We now compare the norm calculated from (A6) with the result of direct integration of the modulus squared of the $m = 0$ pulse $\psi_0(\boldsymbol{r}, t) = \int_0^\infty \mathrm{d}k \mathrm{e}^{-\mathrm{i}kct} \Psi_0(\boldsymbol{r}, k)$. As discussed in appendix B, the beam $\Psi_0(\boldsymbol{r}, k)$ can be expanded in series of products of spherical Bessels and Legendre polynomials, $\Psi_0(\boldsymbol{r}, k) = \sum_0^\infty A_n(k) j_n(kr) P_n(\cos\theta)$. In the evaluation of $\int \mathrm{d}^3r\, |\psi(\boldsymbol{r}, t)|^2$, we use the results ($\eta = \cos\theta$)

$$\int_{-1}^{1} \mathrm{d}\eta\, P_n(\eta) P_m(\eta) = \frac{2\delta_{nm}}{2n+1} \quad \text{and} \quad \int_0^\infty \mathrm{d}r\, r^2\, j_n(kr) j_n(k'r) = \frac{\pi}{2k^2} \delta(k - k')\,. \tag{A 7}$$

The latter formula follows from (A2), since $j_n(\zeta) = \sqrt{(\pi/2\zeta)} J_{n+1/2}(\zeta)$. Thus

$$N = \int \mathrm{d}^3r\, |\psi(\boldsymbol{r}, t)|^2 = 2\pi^2 \sum_0^\infty \frac{1}{2n+1} \int_0^\infty \mathrm{d}k |A_n(k)|^2\, k^{-2}. \tag{A 8}$$

Equivalence of (A8) with (A6) is ensured if

$$\sum_0^\infty \frac{|A_n(k)|^2}{2n+1} = 2k \int_0^k \mathrm{d}\kappa\, \kappa^{-1} \sqrt{k^2 - \kappa^2}\, |f(k,\kappa)|^2. \tag{A 9}$$

It is convenient to define the function $h(k, q) = q\kappa^{-1} f(k, \kappa)$, and to set $q = k\eta$. The sum entering into the norm can be found, when use is made of the Legendre series representation of the Dirac delta function:

$$\sum_0^\infty (2n+1) P_n(\eta) P_n(\eta') = 2\delta(\eta - \eta'), \tag{A 10}$$

$$\sum_0^\infty \frac{|A_n(k)|^2}{2n+1} = k^2 \sum_0^\infty (2n+1) \int_0^1 \mathrm{d}\eta\, h(k,\, k\eta)^* P_n(\eta) \int_0^1 \mathrm{d}\eta'\, h(k,\, k\eta') P_n(\eta')$$

$$= 2k^2 \int_0^1 \mathrm{d}\eta\, |h(k,\, k\eta)|^2 = 2k \int_0^k \mathrm{d}\kappa\, \kappa^{-1} \sqrt{k^2 - \kappa^2}\, |f(k,\kappa)|^2. \tag{A 11}$$

Hence (A 9) is verified, and the expressions for the norm in (A 8) and (A 6) have been proved to be equivalent. For the wave function ψ_0, normalized to unity at the space–time origin, we have

$$f(k,\kappa) = \frac{a^4}{3} k \mathrm{e}^{-ka} \kappa, \quad h(k,q) = q\kappa^{-1} f(k,\kappa) = \frac{a^4}{3} k \mathrm{e}^{-ka} q, \tag{A 12}$$

$$N = (2\pi)^2 \int_0^\infty \mathrm{d}k\, k^{-1} \int_0^k \mathrm{d}q\, |h(k,q)|^2 = \frac{4\pi^2}{9} a^8 \int_0^\infty \mathrm{d}k\, k\, \mathrm{e}^{-2ka} \int_0^k \mathrm{d}q\, q^2 = \frac{\pi^2}{9} a^3. \tag{A 13}$$

The same result follows from direct integration of $N = 2\pi \int_0^\infty \mathrm{d}\rho\, \rho \int_{-\infty}^\infty \mathrm{d}z\, |\psi_0|^2$.

Appendix B. Cylindrical integrals related to spherical sums

If we restrict consideration to beam wave functions which do not depend on the azimuthal angle ϕ, the cylindrically symmetric solutions of the Helmholtz equation may be expressed either as an integral over the cylindrical Bessel function J_0, or as a sum over a product of spherical Bessel functions j_n and Legendre polynomials P_n:

$$\Psi(\boldsymbol{r},k)=\int_0^k \mathrm{d}\kappa\, f(k,\kappa)\mathrm{e}^{\mathrm{i}z\sqrt{k^2-\kappa^2}}J_0(\kappa\rho)=\sum_0^\infty A_n(k)j_n(kr)P_n(\cos\theta). \tag{B 1}$$

(An example of both expressions for the same beam function is provided by the 'proto-beam' in [23], where the coefficients A_n are made explicit.) The corresponding pulse wave functions are then

$$\psi(\boldsymbol{r},t)=\sum_0^\infty P_n(\cos\theta)\int_0^\infty \mathrm{d}k\, A_n(k)\mathrm{e}^{-\mathrm{i}kct}j_n(kr). \tag{B 2}$$

We wish to relate the coefficients $A_n(k)$ of the spherical coordinate expansion to the weight function $f(k,\kappa)$ in the cylindrical coordinate integral, from the equality (B 1). Let us take $\rho=0$ in (B 1), that is, equate the on-axis values. On the axis, we have $r=|z|$, $P_n(\cos\theta)=[\mathrm{sgn}(z)]^n$, so

$$\int_0^k \mathrm{d}\kappa\, f(k,\kappa)\mathrm{e}^{\mathrm{i}z\sqrt{k^2-\kappa^2}}=\sum_0^\infty A_n(k)j_n(k|z|)[\mathrm{sgn}(z)]^n=\sum_0^\infty A_n(k)j_n(kz). \tag{B 3}$$

We change to the variable $q=\sqrt{k^2-\kappa^2}$, with $\kappa d\kappa+qdq=0$, and use the auxiliary function $h(k,q)=q\kappa^{-1}f(k,\kappa)$. The left side of (B 3) becomes $\int_0^k \mathrm{d}q h(k,q)\mathrm{e}^{\mathrm{i}qz}$. We now operate on (B3) with $k\int_{-\infty}^{\infty}\mathrm{d}z\, j_m(kz)$, introduce the dimensionless variables $\eta=q/k$, $\zeta=kz$, and use the known Fourier transform of spherical Bessels ([34], eqn 10.59.1)

$$\int_{-\infty}^{\infty}\mathrm{d}\zeta\, \mathrm{e}^{\mathrm{i}\eta\zeta}\, j_n(\zeta)=\begin{cases}\pi\mathrm{i}^n P_n(\eta), & -1<\eta<1,\\ \dfrac{\pi}{2}(\pm\mathrm{i})^n, & \eta=\pm1,\\ 0, & \pm\eta>1.\end{cases} \tag{B 4}$$

On the right-hand side, we use the orthogonality condition

$$\int_{-\infty}^{\infty}\mathrm{d}\zeta j_m(\zeta)j_n(\zeta)=\frac{\pi\delta_{nm}}{2n+1}. \tag{B 5}$$

The transformed (B 3) evaluates the Bessel–Legendre series coefficients:

$$A_m(k)=\pi(2m+1)\mathrm{i}^m k\int_0^1 \mathrm{d}\eta P_m(\eta)h(k,k\eta). \tag{B 6}$$

For example, when $h(k,q)$ is given by (A 12) formula (B 6) reproduces the proto-beam expansion coefficients found in [23].

References

1. Lekner J. 2017 Energy, momentum, and angular momentum of sound pulses. *J. Acoust. Soc. Am.* **142**, 3428–3435. (doi:10.1121/1.5014058)
2. Whittaker ET. 1904 On an expression of the electromagnetic field due to electrons by means of two scalar potential functions. *Proc. Lond. Math. Soc.* **1**, 367–372. (doi:10.1112/plms/s2-1.1.367)
3. Zangwill A. 2013 *Modern electrodynamics*. Cambridge, UK: Cambridge University Press.
4. Sheppard CJR, Saari P. 2008 Lommel pulses: an analytic form for localized waves of the focus wave mode type with bandlimited spectrum. *Optic. Exp.* **16**, 150–160. (doi:10.1364/OE.16.000150)
5. Zamboni-Rached M. 2009 Unidirectional decomposition method for obtaining exact localized wave solutions totally free of backward components. *Phys. Rev. A* **79**, 013816. (doi:10.1103/PhysRevA.79.013816)
6. Van Enk SJ, Nienhuis G. 1992 Eigenfunction description of laser beams and orbital angular momentum of light. *Opt. Commun.* **94**, 147–158. (doi:10.1016/0030-4018(92)90424-P)
7. Barnett SM, Allen L. 1994 Orbital angular momentum and nonparaxial light beams. *Opt. Commun.* **110**, 670–678. (doi:10.1016/0030-4018(94)90269-0)

8. Lekner J. 2004 Invariants of three kinds of generalized Bessel beams. *J. Opt. A. Pure Appl. Opt.* **6**, 837–843. (doi:10.1088/1464-4258/6/9/004)
9. Bateman H. 1904 The solution of partial differential equations by means of definite integrals. *Proc. Lond. Math. Soc.* **1**, 451–458. (doi:10.1112/plms/s2-1.1.451)
10. Lekner J. 2016 *Theory of reflection*, 2nd edn. Berlin, Germany: Springer.
11. Lekner J. 2001 TM, TE and 'TEM' beam modes: exact solutions and their problems. *J. Opt. A. Pure Appl. Opt.* **3**, 407–412. (doi:10.1088/1464-4258/3/5/314)
12. Trautman A. 1962 Analytic solutions of Lorentz-invariant linear equations. *Proc. R. Soc. Lond. A* **270**, 326–328. (doi:10.1098/rspa.1962.0222)
13. Synge JL. 1960 *Relativity: the general theory*. Amsterdam, The Netherlands: North-Holland.
14. Ziolkowski RW. 1985 Exact solutions of the wave equation with complex source locations. *J. Math. Phys.* **26**, 861–863. (doi:10.1063/1.526579)
15. Donnelly R, Ziolkowski RW. 1993 Designing localized waves. *Proc. R. Soc. Lond. A* **440**, 541–565. (doi:10.1098/rspa.1993.0033)
16. Ziolkowski RW, Besieris IM, Shaarawi AM. 1993 Aperture realizations of exact solutions to homogeneous wave equations. *J. Opt. Soc. Am.* **10**, 75–87. (doi:10.1364/JOSAA.10.000075)
17. Hillion P. 1993 Generalized phases and nondispersive waves. *Acta Appl. Math.* **30**, 35–45. (doi:10.1007/BF00993341)
18. Lekner J. 2004 Localized oscillatory acoustic pulses. *J. Phys. Condens. Matter.* **18**, 3031–3036. (doi:10.1088/0953-8984/18/11/009)
19. Borisov VV, Utkin AB. 2001 Generalization of Brittingham's localized solutions to the wave equation. *Eur. J. Phys.* **21**, 477–480. (doi:10.1007/s100510170155)
20. Kiselev AP. 2003 Generalization of Bateman-Hillion progressive wave and Bessel-Gauss pulse solutions of the wave equation via separation of variables. *J. Phys. A. Math. Gen.* **36**, L345–L349. (doi:10.1088/0305-4470/36/23/103)
21. Lekner J. 2004 Localized electromagnetic pulses with azimuthal dependence. *J. Opt. A. Pure Appl. Opt.* **6**, 711–716. (doi:10.1088/1464-4258/6/7/009)
22. Watson GN. 1944 *Theory of Bessel functions*, 2nd edn. Cambridge, UK: Cambridge University Press.
23. Lekner J. 2016 Tight focusing of light beams: a set of exact solutions. *Proc. R. Soc. A* **472**, 20160538. (doi:10.1098/rspa.2016.0538)
24. Carter WH. 1973 Anomalies in the field of a Gaussian beam near focus. *Opt. Commun.* **64**, 491–495. (doi:10.1016/0030-4018(73)90012-6)
25. Berry MV. 1998 Wave dislocation reactions in non-paraxial Gaussian beams. *J. Mod. Opt.* **45**, 1845–1858. (doi:10.1080/09500349808231706)
26. Nye JF. 1998 Unfolding of higher-order wave dislocations. *J. Opt. Soc. Am.* **15**, 1132–1138. (doi:10.1364/JOSAA.15.001132)
27. Lekner J, Andrejic P. 2018 Nonexistence of exact solutions agreeing with the Gaussian beam on the beam axis or in the focal plane. *Opt. Commun.* **407**, 22–26. (doi:10.1016/j.optcom.2017.08.071)
28. Lekner J. 2004 Angular momentum of electromagnetic pulses. *J. Opt. A. Pure Appl. Opt.* **6**, S128–S133. (doi:10.1088/1464-4258/6/3/021)
29. Lekner J. 2004 Invariants of electromagnetic beams. *J. Opt. A. Pure Appl. Opt.* **6**, 204–209. (doi:10.1088/1464-4258/6/2/008)
30. von Laue M. 1911 Zur Dynamik der Relativitätstheorie. *Ann. Phys.* **35**, 524–542. (doi:10.1002/andp.19113400808)
31. Griffiths DJ. 2011 Resource letter EM-1: electromagnetic momentum. *Am. J. Phys.* **80**, 7–18. (doi:10.1119/1.3641979)
32. Møller C. 1960 *The theory of relativity*. Oxford, UK: Clarendon Press.
33. Lekner J. 2003 Electromagnetic pulses which have a zero momentum frame. *J. Opt. A. Pure Appl. Opt.* **5**, L15–L18. (doi:10.1088/1464-4258/5/4/101)
34. Olver FWJ, Maximon LC. 2010 Bessel functions. In *NIST handbook of mathematical functions* (eds FWJ Olver *et al.*), ch. 10. Cambridge, UK: Cambridge University Press.