Journal of Electrostatics 111 (2021) 103566

Contents lists available at ScienceDirect

Journal of Electrostatics

journal homepage: http://www.elsevier.com/locate/elstat

Polarizabilities of intersecting conducting cylinders

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ARTICLE INFO

Keywords:
Polarizability tensor
Conducting cylinders

ABSTRACT

The longitudinal and transverse polarizabilities of a pair of overlapping conducting parallel cylinders are determined for arbitrary degree of overlap. The polarizabilities of cylinders in contact (vanishing overlap) and those for complete overlap are obtained as special values. Cylinders intersecting at right angles have polarizabilities in accord with those of Palaniappan.

1. Introduction

The problem of the polarizability tensor of a conducting object formed by the intersection of two cylinders has been examined by Radchik et al. [1] for materials characterized by a complex dielectric constant, and by Palaniappan [2,3] for conductors, the latter by the method of images. The image method gives tractable results when the angle of intersection between the cylinders is one of π/n, $n = 2, 3, \ldots$; for example, three images suffice for $n = 2$, cylinders intersecting at right angles. Here we shall remove the restriction on the angle of intersection: the longitudinal and transverse polarizabilities are obtained at all angles of intersection. The results are restricted to overlapping cylinders of equal radius a. Their axes are parallel to the z axis. The methods are similar to those used by the author in obtaining the electrostatic properties of cylinders in an external field [4–7].

We shall use a form of bicylindrical coordinates u, v related to the x, y coordinates by the conformal transformation

$$v + iu = \ln\frac{x + i\ell - iy}{x - i\ell - iy} \quad \text{or} \quad x + iy = i\ell \coth\frac{v + iu}{2} \tag{1}$$

Equating the real and imaginary parts of (1) gives

$$\frac{x}{\ell} = \frac{\sin u}{\cosh v - \cos u}, \quad \frac{y}{\ell} = \frac{\sinh v}{\cosh v - \cos u} \tag{2}$$

The ranges of u, v are $-\pi \le u \le \pi$, $-\infty < v < \infty$. The origin on the xy plane corresponds to $u = \pm\pi, v = 0$. The distance from the origin is $\rho = \sqrt{x^2 + y^2} = \ell\sqrt{\frac{\cosh v + \cos u}{\cosh v - \cos u}}$. Thus large ρ/ℓ obtains when u, v both tend to zero; then $\rho \to 2\ell/\sqrt{u^2 + v^2}$.

Elimination of v from equation (2) gives circles with centers on the x axis:

$$(x - \ell \cot u)^2 + y^2 = \frac{\ell^2}{\sin^2 u} \tag{3}$$

These circles all meet at the points $[0, \pm\ell]$. The value $u = u_a$ corresponds to a circular cylinder parallel to the z axis, centered on $[0, \ell \cot u_a]$, with radius $a = \ell/\sin u_a$. Likewise the circular cylinder $u = -u_b$ $(u_b > 0)$ is centered on $[0, -\ell \cot u_b]$, and has radius $b = \ell/\sin u_b$. In the following we specialize to $b = a$, $u_b = u_a$.

Elimination of u from equation (2) gives circles with centers on the y axis:

$$x^2 + (y - \ell \coth v)^2 = \frac{\ell^2}{\sinh^2 v} \tag{4}$$

Thus constant v corresponds to circular cylinders parallel to the z axis, centered on $[0, \ell \coth v]$, with radius $a = \ell/\sinh v$. These cylinders are orthogonal to those represented by (3).

The scale length ℓ is determined once we specify the distance between the $u = u_a$ and $u = -u_a$ cylinder axes, which we will call c. We have

$$c = 2\ell \cot u_a = 2a \cos u_a = 2\sqrt{a^2 - \ell^2}, \quad \ell^2 = a^2 - (c/2)^2 \tag{5}$$

Also from (5) and $a = \frac{\ell}{\sin u_a}$, u_a and $U = 2u_a$ may be explicitly expressed in terms of the cylinder radii a, a and center to center distance c:

$$\cos u_a = \frac{c}{2a}, \quad \cos U = \frac{c^2 - 2a^2}{2a^2}, \quad \ell = \frac{a^2}{c}\sin U \tag{6}$$

The triangle with sides a, a, c containing the point of intersection of the two cylinders (see Fig. 1) has angles u_a at the axes of the cylinders, and angle $\pi - 2u_a = \pi - U$ at the line of intersection. The height of the triangle is ℓ: the cylinders intersect on lines parallel to the z axis passing

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https://doi.org/10.1016/j.elstat.2021.103566
Received 14 January 2021; Received in revised form 11 February 2021; Accepted 15 February 2021
Available online 12 March 2021
0304-3886/

through $[0, \pm\ell]$. The angle of intersection of the cylinder surfaces is $U = 2u_a$. The scale length $\ell = [a^2 - (c/2)^2]^{1/2}$ is the distance from the z axiz to the apex of the triangle, and thus ℓ is the radius of the circle of intersection of the two cylinders. The limit of touching cylinders is $c \to 2a,\ \ell \to 0,\ u_a \to 0$. The opposite limit is that of completely overlapping cylinders, $c \to 0, \ell \to a, u_a \to \pi/2$.

2. Intersecting cylinders in longitudinal external field

The Laplace equation $(\partial_x^2 + \partial_y^2)V = 0$ is satisfied by any differentiable function of $v + iu$; the choice $e^{-\kappa(v+iu)}$ gives, for example, the functional form $\cosh \kappa v \sin \kappa u$, which may be integrated over the separation constant κ after multiplication by an amplitude $A(\kappa)$. The method used here follows that of Radchik et al. [1]. We need to satisfy the boundary conditions that $V(\pm u_a, v) = 0$ (we may choose the potential on the cylinders to be zero), and that $V \to -E_0 x$ far from the cylinders, that is to the potential of the external field directed along the x direction.

To find the *longitudinal polarizability* of the composite body, we construct a potential which tends to $-E_0 x$ far from the body, and which is constant on the body. The electric potential in a longitudinal field is odd in x, and even in y if it is zero on $y = 0$; we write it as

$$V(u, v) = -E_0 x + E_0 \ell \int_0^\infty d\kappa A(\kappa, u_a) \sinh \kappa u \cos \kappa v \tag{7}$$

The values of x/ℓ on $u = \pm u_a$ are $\pm \sin u_a/(\cosh v - \cos u_a)$. Since the potential on the body composed of the intersecting cylinders is chosen to be zero, we need the following equality to hold at all v:

$$\int_0^\infty d\kappa A(\kappa, u_a) \sinh \kappa u_a \cos \kappa v = \frac{x_a}{\ell} = \frac{\sin u_a}{(\cosh v - \cos u_a)} \tag{8}$$

Expansion of the defining relation (1) in powers of e^{-v} when $v > 0$ gives us

$$\frac{x}{\ell} = 2\sum_1^\infty e^{-nv} \sin nu, \quad \frac{y}{\ell} = 1 + 2\sum_1^\infty e^{-nv} \cos nu \quad (v > 0) \tag{9}$$

The amplitude function $A(\kappa, u_a)$ is given by the inverse cosine transform of (8):

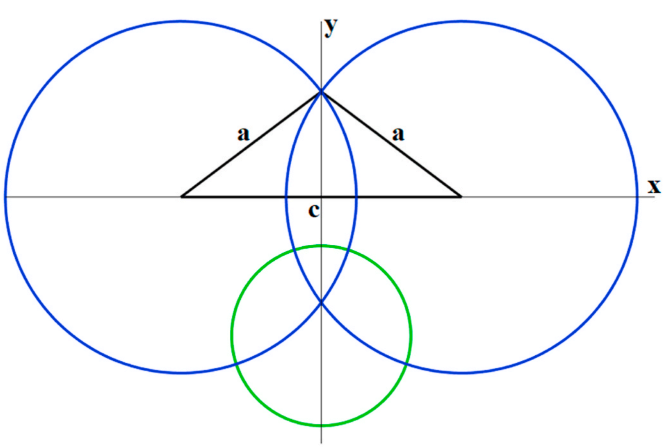

Fig. 1. Intersecting cylinders of radii a, a (blue) with distance between their axes c. Also shown is one of the family of cylinders orthogonal to the intersecting pair, in green. The blue cylinder surfaces correspond to $u = \pm u_a$, the green cylinder to a particular value of v. The figure is drawn for $a : \frac{c}{2} : \ell = 5:4:3$; the cylinders intersect at angle $2u_a = 2 \arcsin 3/5 \approx 73.74°$. The external field will be referred to as *longitudinal* when it is horizontal in the Figure, and *transverse* when vertical. (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

$$\begin{aligned} A(\kappa, u_a) \sinh \kappa u_a &= \frac{2}{\pi} \int_0^\infty dv \cos \kappa v \frac{x_a}{\ell} = \frac{4}{\pi} \int_0^\infty dv \cos \kappa v \sum_1^\infty e^{-nv} \sin nu_a \\ &= \frac{4}{\pi} \sum_1^\infty n \frac{\sin nu_a}{n^2 + \kappa^2} = 2\frac{\sinh \kappa(\pi - u_a)}{\sinh \kappa\pi} \end{aligned} \tag{10}$$

(For the sum, see the methods of Section 125 of Bromwich [8].) The potential (7) is thus explicitly

$$V(u, v) = -E_0 x + 2E_0 \ell \int_0^\infty d\kappa \frac{\sinh \kappa(\pi - u_a) \sinh \kappa u}{\sinh \kappa\pi \sinh \kappa u_a} \cos \kappa v \tag{11}$$

From (1) we see $\rho^2 = x^2 + y^2 = \ell^2 \frac{\cosh v + \cos u}{\cosh v - \cos u} \to \frac{4\ell^2}{u^2 + v^2}$ for small $u^2 + v^2$. Large distance from the origin corresponds to small u, v. Hence the asymptotic form of (11) is

$$V(u, v) \to -E_0 x + 2E_0 \ell\, u \int_0^\infty d\kappa\, \kappa \frac{\sinh \kappa(\pi - u_a)}{\cosh \kappa\pi \sinh \kappa u_a} \tag{12}$$

When ρ is very large compared to the radius of the two intersecting cylinders, the potential of the polarized body tends to $2xp_L/\rho^2$. Here p_L is the dipole moment per unit length of the composite body, in this case directed longitudinally (along the x direction). The factor 2 comes from the definition of a two-dimensional dipole moment as the limit of two parallel line charges of opposite sign approaching each other [4]. From (1), $2x/\rho^2 \to u/\ell$ when ρ is large and both u and v are small. The leading term at large ρ due to the polarization of the body is thus, with $p_L = \alpha_L E_0$,

$$2E_0 \ell\, u \int_0^\infty d\kappa\, \kappa \frac{\sinh \kappa(\pi - u_a)}{\cosh \kappa\pi \sinh \kappa u_a} = \frac{E_0 \alpha_L\, u}{\ell} \tag{13}$$

The longitudinal polarizability per unit length of the body composed of two intersecting cylinders of equal radii is therefore

$$\begin{aligned} \alpha_L &= 2\ell^2 \int_0^\infty d\kappa\, \kappa \frac{\sinh \kappa(\pi - u_a)}{\cosh \kappa\pi \sinh \kappa u_a} = 2\ell^2 \int_0^\infty d\kappa\, \kappa \{\coth \kappa u_a - \coth \kappa\pi\} \\ &= \frac{2\ell^2}{\pi^2} \int_0^\infty dt\, t \left\{\coth \frac{u_a}{\pi} t - \coth t\right\} = \frac{\ell^2}{6}\left[\left(\frac{\pi}{u_a}\right)^2 - 1\right] \end{aligned} \tag{14}$$

To obtain the final result we have expanded the hyperbolic cotangents, for example $\coth t = 1 + 2\sum_1^\infty e^{-2nt}$, integrated over t, and used the sum $\sum_1^\infty n^{-2} = \pi^2/6$.

Limiting polarizability values are at $c \to 2a, u_a \to 0$ (cylinders in contact), and $c \to 0, u_a \to \pi/2$ (complete overlap). Since $\ell = a \sin u_a$, the contact value is $\alpha_L = (\pi^2/6)a^2$, in agreement with the result previously obtained [4]. For complete overlap, $u_a \to \pi/2,\ \ell \to a, \alpha_L \to a^2/2$, the expected polarizability of a solitary cylinder [4]. When $u_a = \pi/3$ we find $\alpha_L = a^2$. Rational multiples of a^2 are obtained when $u_a = \pi/n$ and $\sin^2 \pi/n$ is rational.

For equal cylinders intersecting at right angles, $c = \sqrt{2}a, \ell = a/\sqrt{2}$, $u_a = \pi/4$, and the expression (14) agrees with the value obtained by setting $a = b,\ c = \sqrt{2}a$ in the $n = 2$ form of equation (4.4) of Palaniappan [3], namely $\alpha_L = \frac{5}{4}a^2$. Note that the Palaniappan expression is for $D_L = 2\alpha_L$; the factor 2 was discussed above equation (13).

3. The transverse polarizability

For the intersecting cylinders in a transverse field the external field points along the y axis, tending to $-E_0 y$. The solution of Laplace's equation representing the potential is now to be odd in y and even in x:

$$V(u,v) = -E_0 y + E_0 \ell \int_0^\infty d\kappa B(\kappa,u_a)\cosh\kappa u \sin\kappa v \tag{15}$$

The potential is again chosen to be zero on the overlapping cylinders, to be satisfied by

$$\int_0^\infty d\kappa B(\kappa,u_a)\cosh\kappa u_a \sin\kappa v = \frac{y_a}{\ell} = \frac{\sinh v}{(\cosh v - \cos u_a)} \tag{16}$$

The amplitude function $B(\kappa,u_a)$ is given by the inverse sine transform of (16):

$$B(\kappa,u_a)\cosh\kappa u_a = \frac{2}{\pi}\int_0^\infty dv \sin\kappa v \frac{y_a}{\ell} = \frac{2}{\pi}\int_0^\infty dv \sin\kappa v \left\{1 + 2\sum_1^\infty e^{-nv}\cos nu_a\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\kappa} + 2\kappa\sum_1^\infty \frac{\cos nu_a}{n^2+\kappa^2}\right\} = 2\frac{\cosh\kappa(\pi-u_a)}{\sinh\kappa\pi} \tag{17}$$

We have used the representation (9) for y/ℓ. The potential (15) is thus explicitly

$$V(u,v) = -E_0 y + 2E_0\ell\int_0^\infty d\kappa \frac{\cosh\kappa(\pi-u_a)\cosh\kappa u}{\sinh\kappa\pi\cosh\kappa u_a}\sin\kappa v \tag{18}$$

Far from the two intersecting cylinders, the potential of the polarized body tends to $2yp_T/\rho^2$, where $p_T = \alpha_T E_0$ is the dipole moment of the body, now directed transversely (along the y direction). From (2), $2y/\rho^2 \to v/\ell$ when ρ is large and both u and v are small. The leading term due to the polarization of the body is thus,

$$2E_0\ell v\int_0^\infty d\kappa\kappa\frac{\cosh\kappa(\pi-u_a)}{\sinh\kappa\pi\cosh\kappa u_a} = \frac{E_0\alpha_T v}{\ell} \tag{19}$$

The transverse polarizability of the body composed of two intersecting cylinders of equal radii is thus

$$\alpha_T = 2\ell^2\int_0^\infty d\kappa\,\kappa\frac{\cosh\kappa(\pi-u_a)}{\sinh\kappa\pi\cosh\kappa u_a} = 2\ell^2\int_0^\infty d\kappa\,\kappa\{\coth\kappa\pi - \tanh\kappa u_a\}$$

$$= \frac{2\ell^2}{\pi^2}\int_0^\infty dt\,t\left\{\coth t - \tanh\frac{u_a}{\pi}t\right\} = \frac{\ell^2}{12}\left[\left(\frac{\pi}{u_a}\right)^2 + 2\right] \tag{20}$$

Since $\ell = a\sin u_a$, the contact value $u_a \to 0$ is $\alpha_T = (\pi^2/12)a^2$, in agreement with the result previously obtained [4]. For complete overlap, $u_a \to \pi/2$, $\ell \to a$, $\alpha_T \to a^2/2$, the expected polarizability of a solitary cylinder [4]. Rational values are obtained when $u_a = \pi/n$ and $\sin^2\pi/n$ is rational, as in the longitudinal case. For cylinders intersecting at right angles we have $\ell = a/\sqrt{2}$, $u_a = \pi/4$, and the expression (20) agrees with the value obtained by setting $a = b$, $c = \sqrt{2}a$ in equation (4.2) of Palaniappan, namely $\alpha_T = \frac{3}{4}a^2$.

Fig. 2 shows the variation of the polarizabilities with the amount of overlap of the two cylinders, from contact (no overlap) at $U = 0$ to complete overlap at $U = \pi$.

4. Summary and discussion

Palaniappan [2,3] has obtained the polarizability tensor components for overlapping cylinders by the method of images, when the angle of intersection of the cylinders is π/n, for integer n, at which the method of images gives algebraic values for the polarizabilities. We obtain algebraic expressions at any angle of intersection, which we may rewrite explicitly in terms of the lengths in the problem as

$$\alpha_L = \frac{a^2 - c^2/4}{6}\left[\left(\frac{\pi}{\arccos\frac{c}{2a}}\right)^2 - 1\right], \alpha_T = \frac{a^2 - c^2/4}{12}\left[\left(\frac{\pi}{\arccos\frac{c}{2a}}\right)^2 + 2\right] \tag{21}$$

There is smooth decrease in both α_L and α_T from their contact values as the degree of overlap increases. At complete overlap both polarizabilities correctly reach the polarizability of a single cylinder. When the cylinders intersect at right angles, the general formulae agree with the Palaniappan expressions.

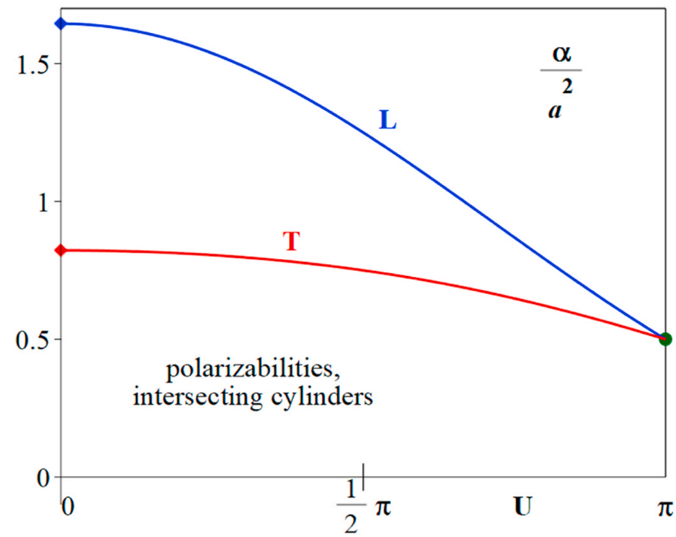

Fig. 2. Two-cylinder longitudinal and transverse polarizability per unit length. The figure shows variation with overlap, as determined by the angle of intersection $U = 2u_a$ of the two cylinders. U is zero at contact (no overlap), $U = \pi$ at complete overlap. The diamonds at left are at the contact values $\pi^2/6 \approx 1.645$ of α_L/a^2 and $\pi^2/12 \approx 0.822$ of α_T/a^2. The circle at right is the single-cylinder value $\alpha = a^2/2$ for complete overlap.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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